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NUMERICAL SOLUTIONS OF SHORT CANTILEVER BEAMS SUBJECTED TO END MOMENT

M Iftekher Rahman, M Zubaer Hossain, S Reaz Ahmed and M Wahhaj Uddin

Department of Mechanical Engineering, Bangladesh University of Engineering & Technology, Dhaka-1000, Bangladesh.

ABSTRACT
Finite-difference analysis of stresses in short cantilever beams subjected to end moment is presented in this paper. Displacement potential function formulation has been used in conjunction with finite-difference method of solution to analyze the state of stresses in the beam having different aspect ratio. The corresponding finite-element solutions for the present beam problem have been generated by using the commercial finite element code, ANSYS. The results are presented in the form of graphs and are compared with the analytical solutions of simple beam theory. The comparative analysis shows that the solutions for stresses differ significantly around the fixed support of the beams, the most critical section in terms of stresses; however, away from the end solutions agree reasonably well.

Keywords: Stress analysis, finite-difference method, finite-element method, short cantilever beam.

1. INTRODUCTION
The use of built-in cantilevers in the construction of engineering structures is quite extensive. Although the simple flexure theory is widely used for the design purposes of such beams, it is simply inadequate to give information regarding local stresses around the regions of the loads and supports of beams. Moreover, when the beam depth becomes comparable to its length, design based on classical theory becomes unreliable and uneconomic, since the distributions of bending and shearing stresses are not linear and parabolic, respectively, in cases of deep / short beams [1].

As far as literature is concerned several attempts have been made for the analysis of deep beams. However, most of the researchers concentrated on the solution of the simply supported beams [1-4]. Successful attempts towards the prediction of reliable stresses around the fixed end of short cantilevers have hardly been made in the past mainly because of the inability to satisfy the actual conditions of the supports in a justified manner, by their mathematical formulation. It is however noted that Murty [5] obtained the solution of a tip-loaded cantilever beam by using his higher order theory for bending. However, the solution gives a constant shearing stress distribution around the fixed end and the classical parabolic distribution for sections sufficiently away from the fixed end. It has been found that the results do not always satisfy the restraint conditions as well as the conditions of equilibrium. Moreover, the shearing stresses along the upper and lower edges do not vanish; rather their values become large in the neighborhood of the fixed support. Also, the strain energy contributed by bending stress is neglected in his formulation, and the Poisson's ratio is assumed to be zero.

Taking into account all the limitations, the potential displacement function formulation [6] has been used in conjunction with the finite-difference method of solution to predict the displacement as well as stress distributions in short cantilever beam subjected to a pure bending moment at its free end. The accuracy as well as reliability of the finite-difference computer code has been verified repeatedly through the application of the program to a number of practical problems of engineering [7-12]. The present beam problem has been analyzed extensively where three different values of length-to-depth ratios are considered. Finally, an attempt is made to obtain the corresponding solutions using the finite element software, ANSYS and simple beam theory and the results are compared with those obtained by the present finite-difference method of solution.

2. DESCRIPTION OF THE BEAM PROBLEM
The geometry and loading configuration along with the FDM mesh are illustrated in Fig. 1. The present problem is a typical fixed ended cantilever beam of uniform rectangular cross section, which is subjected to pure bending moment at its free end. The physical conditions at the four boundaries of the beam when stated mathematically are as follows:

For the top surface, which is a free boundary, the normal and tangential components of stress are:

$$\sigma_x(x,y)=\sigma_{xy}(x,y)=0 \;\;,\;\; \text{for } \; 0\le y\le \text{L} \;\;,\;\; \text{x}=0$$

For the bottom surface, which is also a free a boundary, the normal and tangential components of stresses are:

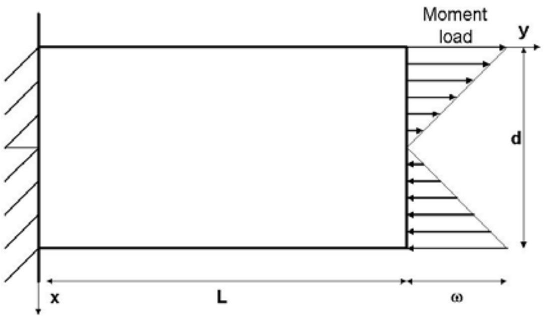


Fig. 1(a) Geometry and loading for the Cantilever beam

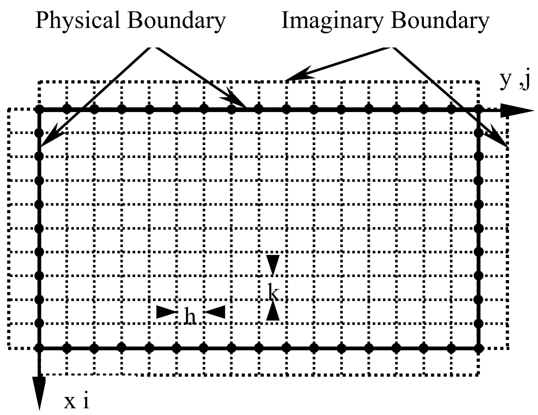


Fig. 1(b): Finite-difference discretization of the beam body.

$\sigma_y(x,y)=\sigma_{xy}(x,y)=0$, for $0\le y\le L$, $x=d$

At the left lateral end, constraint to have zero slope, the respective normal and tangential components of displacement are

$u_x(x,y)=u_y(x,y)=0$, for $0\le x\le L$, $y=0$

For the right lateral end, which is subjected to a pure bending moment, the corresponding normal and tangential stresses are,

$\sigma_x(x,y)=\frac{y}{d/2}KN/mm^2$; $\sigma_{xy}(x,y)=0$, for $0\le x\le 0$, $y=0$

3. SOLUTION OF SIMPLE BEAM THEORY

The stress caused by the bending moment is known as the bending or flexure stress, which is expressed by the following formula:

$$\sigma_x = \frac{M\times y}{I} = \frac{\omega}{b}\left(\frac{y}{d/2}\right) \quad [1]$$

where, M, I and y are bending moment, moment of inertia and distance from neutral axis, respectively. One of the important assumptions on which this simple flexure theory is based is that the plane sections of the beam remain plane after deformation. The relation between the shearing stress (σ_{xy}) and the vertical shear force, for a rectangular cross section is given by,

$$\sigma_{xy} = \frac{V}{Ib}\left(\frac{(d)^2}{4}-y^2\right), \quad [2]$$

where, V, b, 2c are shear force, beam thickness and beam depth, respectively. From the above mathematical expressions, it is obvious that, (1) the flexure stress at any section varies linearly with the distance from the neutral axis; maximum bending stress occurs at the top and bottom surfaces, and (2) shearing stress is distributed parabolically across the depth of the section; maximum shearing stress occurs at the neutral axis.

4. FINITE-DIFFERENCE SOLUTION

Finite-difference solutions are obtained using displacement potential function formulation, where the problem has been formulated in terms of a single potential function, $\psi(x,y)$ defined in terms of the two displacement components, u_x and u_y. The governing differential equation for the solution of two-dimensional elastic problem is given by [6],

$$\frac{\partial^4\psi}{\partial x^4}+2\frac{\partial^4\psi}{\partial x^2\partial y^2}+\frac{\partial^4\psi}{\partial y^4}=0 \quad [3]$$

The boundary conditions in terms of the function ψ are given by [10],

$$u_x(x,y)=\frac{\partial^2\psi}{\partial x\,\partial y} \quad [4]$$

$$u_y(x,y)=-\frac{1}{1+\mu}\left[2\frac{\partial^2\psi}{\partial x^2}+(1-\mu)\frac{\partial^2\psi}{\partial y^2}\right] \quad [5]$$

$$\sigma_{xx}(x,y)=\frac{E}{(1+\mu)^2}\left[\frac{\partial^3\psi}{\partial x^2\partial y}-\mu\frac{\partial^3\psi}{\partial y^3}\right] \quad [6]$$

$$\sigma_{yy}(x,y)=-\frac{E}{(1+\mu)^2}\left[(2+\mu)\frac{\partial^3\psi}{\partial x^2\partial y}+\frac{\partial^3\psi}{\partial y^3}\right] \quad [7]$$

$$\sigma_{xy}(x,y)=\frac{E}{(1+\mu)^2}\left[-\frac{\partial^3\psi}{\partial x^3}+\mu\frac{\partial^3\psi}{\partial x\partial y^2}\right] \quad [8]$$

In the present approach, the whole problem has been formulated in such a way that single function ψ has to be evaluated from the bi-harmonic equation (3) associated with the boundary condition (4-8) that are specified at the bounding edges of the beam. The essential feature of the numerical version of the formulation is that the original governing differential equations of the boundary-value problem is replaced by a finite set of simultaneous algebraic equations and the solution of this simultaneous algebraic equations provide us with an approximation for the displacement and stress within the beam. For the present numerical calculation, Modulus of Elasticity and Poisson's ratio are taken as 209 GPa and 0.3, respectively.

5. FINITE ELEMENT SOLUTION

Finite element solutions have been obtained using the standard elastic facilities available in the ANSYS software. Plane stress assumption is used here because, in practice, the thickness of such a beam is likely to be small when compared with the length or depth [1].

For this analysis 2-D structural element (PLANE 82) is used because of its regular rectangular shape, which makes it convenient to produce data at different sections of interest. The element is defined by 8 nodes having two degrees of freedom at each node, transition in x and y directions.

For the present numerical calculation, the total number of elements taken is 100 for the beams having L/d ratio of 1, 2 and 3. In order to maintain a similar level of accuracy in the solutions, it was desired to keep the element numbers the same for all the beams.

6. RESULTS AND DISCUSSION

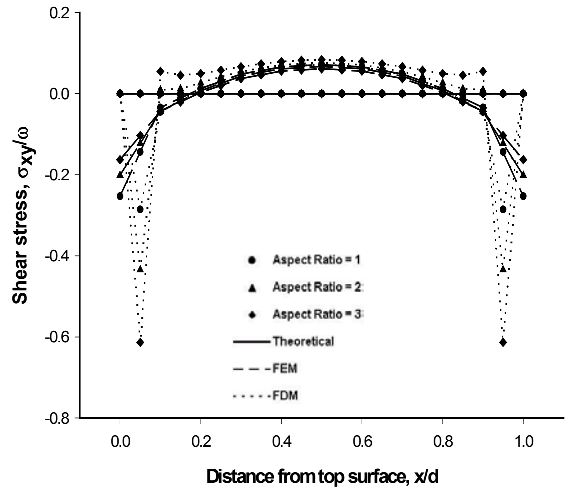
In the present section, numerical solutions for stresses and displacements obtained by the finite-difference approach are analyzed in details, and the results are compared with the finite-element and analytical solutions, at different transverse sections of the beams.

Figure 2 describes the comparison of three different solutions for the bending, shearing and normal stresses at the fixed support of the beams having different aspect ratios. In contrast with the typical parabolic distribution of shearing stress along the beam depth, the beam is virtually free from shearing stress as the beam is loaded by a pure bending moment at its end. That is why the shearing stress given by simple theory at different transverse sections ultimately reduced to a single horizontal line passing through the origin, as shown in fig. 2(a). The theoretical results of bending stress vary linearly along the beam depth, and the stress level is identical at all the transverse sections including the fixed support (fig. 2(b)). This is due to the fact that the applied constant bending moment is acting throughout the entire span of the beam. Moreover, the simple theory tells us that the stresses in the present beam are completely independent of the length-to-depth ratios.

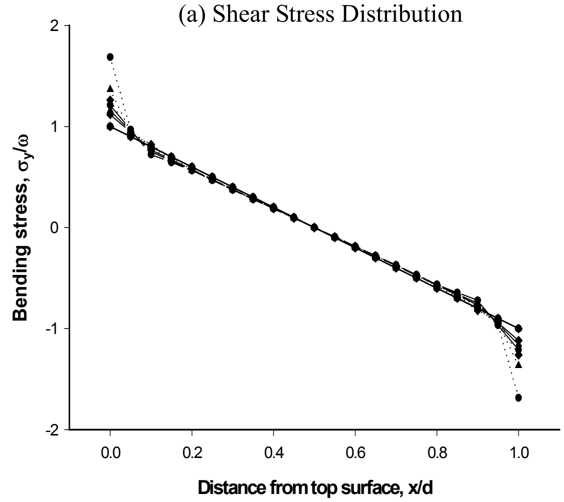
The corresponding finite-different solutions for the beam differ quite significantly from those of simple theory, especially, in the neighborhood of the fixed support of the beam. However, this discrepancy is found to decrease as we move away from the fixed end, where the numerical solutions compare well with the analytical solution of the simple theory.

From figure 2(a), it is observed that the distribution of shearing stress differs significantly from that obtained by simple theory, at the fixed support of the beam, which is theoretically free from shearing stress. However, at the top and bottom surfaces of the beam the finite-difference solutions of shearing stress match exactly with the theoretical results. From the comparisons of solutions of FEM and FDM, it is evident that both the results compare well except at the top and bottom surface of the beams. FDM solutions of shear stress at top and bottom surface are in accordance with the simple beam theory. Surface shear stresses at each section are zero. But FEM provides nonzero solutions both in top and bottom surface. This is perhaps due to the general limitation of the FEM in predicting the surface stresses, which has however been reported by several researchers [13-14]. Therefore, the finite element prediction of shearing stress, especially at the two surfaces of the beams, are not reliable and the discrepancy is found to be more pronounced near the vicinity of the fixed end, where FDM provides accurate solutions.

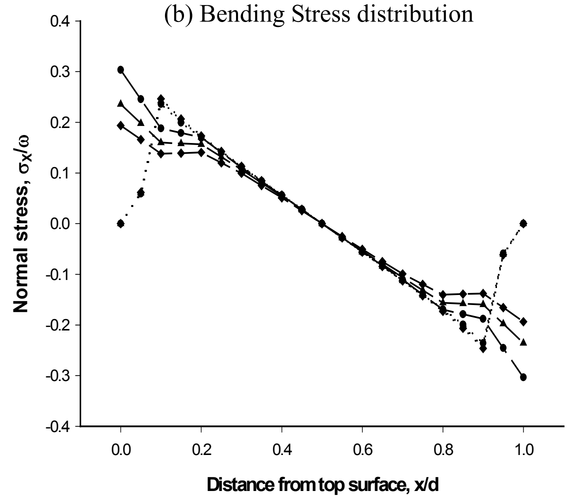
Bending stress distribution at the fixed end (Fig. 2(b)) is almost linear, differing mainly at the two corner points, which are in general the points of singularity. The present results show that the maximum bending stress at the



(a) Shear Stress Distribution



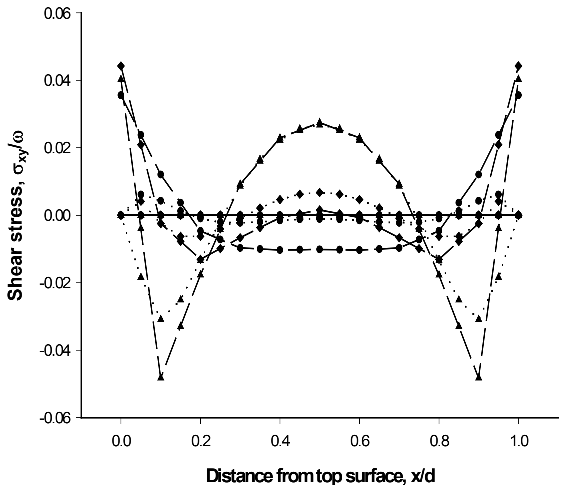
(b) Bending Stress distribution



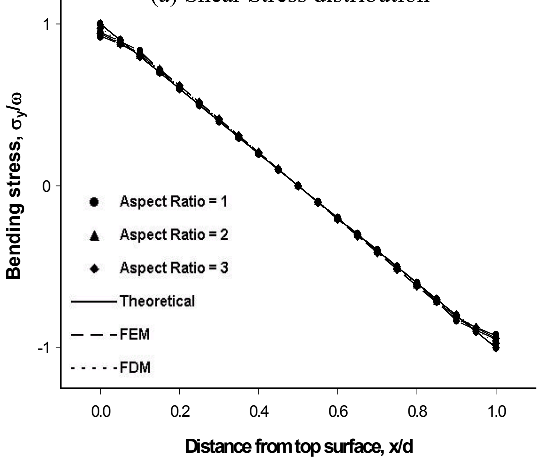
(c) Normal Stress distribution

Fig. 2 Comparison of different solutions for stresses at fixed end of the beams

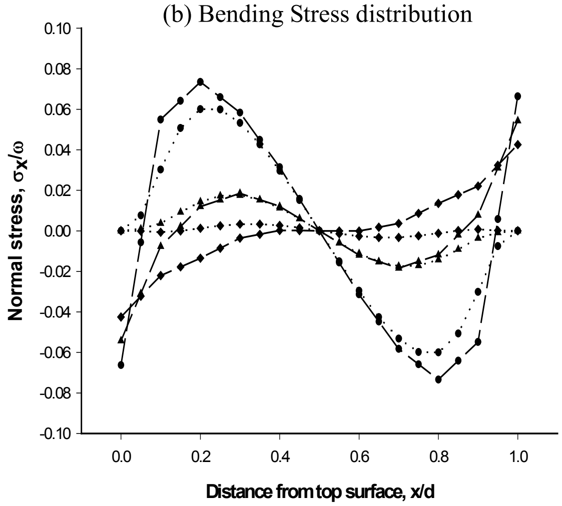
upper and bottom surfaces of the support are higher than those of all other transverse sections of the beam. Except the fixed end, all the solutions are



(a) Shear Stress distribution



(b) Bending Stress distribution



(c) Normal Stress distribution

Fig.3: Comparison of different solutions for stresses at the transverse section, y/L=0.1 of the beams

identical for all the sections of the beam and are in good agreement with the theoretical solution.

In addition to bending and shearing stresses, finite-difference solutions for the distribution of normal stress component, σ_x are obtained and those at the fixed end are presented in Figure 2(c). Note that simple beam theory does not take care of this stress component, although the importance of this stress component cannot be neglected as far as short beams are concerned. The normal stress component varies almost linearly along the fixed end of the beam, keeping the lower half portion under compression and the upper half portion under tension. From FEM solution, magnitudically, the normal stress component is even more significant than the shearing stress developed at the fixed support. The comparison of present finite difference solution with that of FEM, at the fixed support, shows that the two solutions agree well only around the mid-region of the support. However, FEM predictions of normal stress σ_x around the two corners of the support are found to be highly unreliable, as they do not conform to the requirement of the physical characteristics of the beam support. It is noted that the present numerical solution satisfies all the requirements exactly, at or away from the two corners of the support.

In order to investigate the influence of slenderness ratio on the solution, results are obtained for different aspect ratios (L/d). Figure 2 & 3 represent the distribution of shear, bending and normal stresses obtained by FDM, FEM and simple beam theory at the fixed end and at the transverse section, y/d = 0.10 respectively for three different aspect ratios.

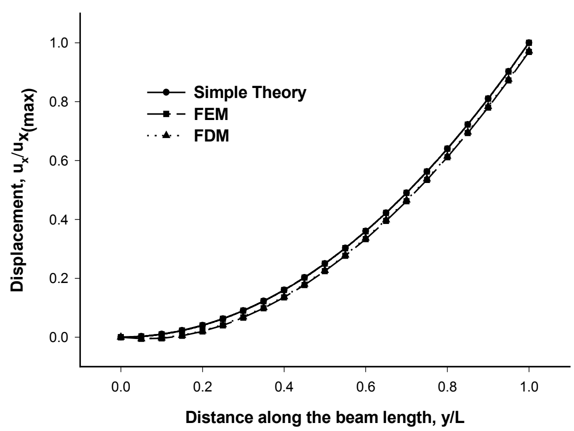
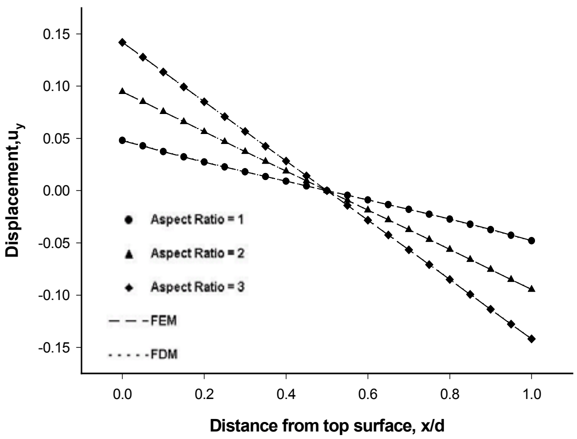
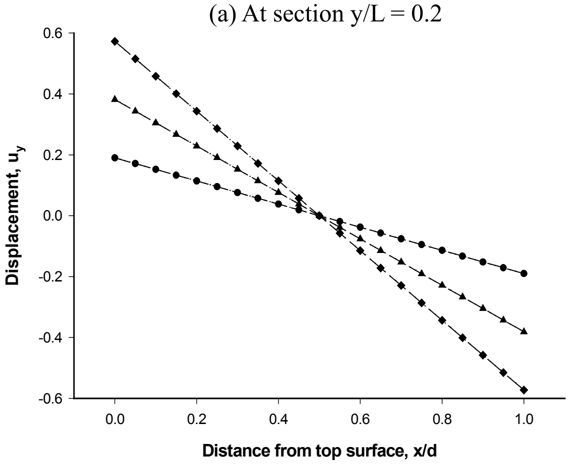


Fig. 4: Distribution of displacement component u_x along the neutral axis of the beam, L/d = 1.

From fig. 2(a) and 3(a), it is observed that both the FDM and FEM solutions of shear stress are not independent of beam aspect ratio, rather maximum shearing stress increases with the increase of aspect ratio, especially for sections around the fixed support. For example, at the fixed end, y/L = 0.0, the values of the normalized shearing stress are -0.3, -0.42 and -0.61 for L/d = 1, 2 and 3, respectively. It can be seen that at a considerable distance from the fixed end, y/L = 0.1, FEM solutions still differ from those of FDM as the FEM predictions at the surfaces are not zero. But the dependency on the aspect ratio is found to be reduced significantly at sections away from fixed end. Bending stress distributions are almost identical for FDM, FEM and for simple theory for all the L/d ratios inspected. The non-linearity is observed mainly around the two corner zones of the support. But this non-linearity diminishes as distance from the fixed end and aspect ratio is increased. Nearly identical variation of the normal stress component, as predicted by FDM, is observed along the support of all the beams inspected. FEM solutions, however, show dependency of the normal stress on aspect ratio, mainly at the two corner zones of the support.



(a) At section y/L = 0.2



(b) At section y/L = 0.8

Fig. 5: Distribution of Displacement component u_y at different transverse section of the beams

Figure 4 demonstrates the deformation pattern of the neutral axis of the beam when subjected to a pure bending moment at its free end. Theoretical results are predicted using the equation

$$u_x = \frac{My^2}{2EI} \qquad [9]$$

Typical deflection characteristics of a cantilever beam is confirmed here through the finite-difference as well as Finite Element prediction of displace component, as maximum deflection occurs at its free end, while the fixed support is free from any displacement.

Figure 5 represents the y-component of displacement u_y (Dimension in mm) at two different sections of the beams. There is no mathematical formulation in simple theory to predict u_y. Using finite-difference method of solution y-component of displacement can be determined accurately. For the current beam problem, the comparative analysis of displacements predicted by FDM and FEM shows that the two numerical solutions are in excellent agreement with each other at each and every section of the beams.

7. CONCLUSIONS

The solution of short built-in cantilever beam had rarely been attempted in the past mainly because of the inability of satisfying the actual conditions of the fixed support of the beam in a justifiable manner. The fixed support of a cantilever beam, subjected to a pure bending moment at its free end, has been verified to be the most critical section in terms of stresses. The present investigation shows that the finite-difference predictions of bending and shear stresses compare reasonably well with the results of simple beam theory, especially for sections away from the fixed end. For predicting the critical stresses at the support, the simple theory has been verified to be unreliable for design purposes of such beams, no matte what value of length-to-depth ratio is considered. Furthermore, the comparison with FEM solutions of displacements and stresses reveals that the two solutions are in good agreement with each other, except at the critical corner zones of the support. More specifically, the FEM prediction of stresses at the corner zones of the fixed support are not very accurate and thus are not reliable.

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NOMENCLATURE

Symbol	Meaning
x, y	Rectangular Coordinate
u_x, u_y	Displacement components
σ_x	Normal Stress
σ_y	Bending Stress
σ_{xy}	Shear Stress
L	Beam Length
d	Beam Depth
b	Beam Thickness
ψ	Displacement Potential Function
M	Moment
E	Modulus of Elasticity
I	Moment of Inertia
ω	Intensity of Moment Load
V	Shear Force
μ	Poisson’s ratio