

Large Deflection Analysis of the Superelastic Shape Memory Alloy Beams

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ABSTRACT

Because of the peculiar stress induced martensitic transformations (SIMT) phenomena, the superelastic shape memory alloy (SMA) can recover too large strains upon unloading unlike the traditional engineering materials. Consequently, known as a functional material, SMA is typically used as a slender beam to undergo large deflections in the modern adaptive structures. For such cases, a theoretical calculation based on small deflection theory is likely to yield inaccurate results particularly for slender SMA beams at high intensity loads. The large deflection theory, on the other hand, invariably involves nonlinear equations having no closed form solutions. That in turn leads to the different numerical techniques for solving those equations. Observing these facts, a computer code based on 'C' has been developed using two different numerical techniques, for analyzing the behaviors of the superelastic shape memory alloy (SMA) beams. The load-deformation curves of the beams have been predicted by using the developed code exploiting the nonlinear as well as the classical linear theories of the beams. Experiments of the beams with fixed-free end condition are also performed to verify the numerical results and check the soundness of the computer code. Among others, the results show that the SMA beam's bending characteristics change remarkably with increasing loading.

Keywords: Shape Memory Effect, Shape Memory Alloy, Large Deflection Beam Theory

1. INTRODUCTION

As far as the peculiar stress induced martensitic transformations (SIMT) is concerned, the superelastic shape memory alloy (SMA) can recover too large strains upon unloading unlike the traditional engineering materials [1-8]. Consequently, superelastic SMA is typically used as a slender beam to undergo large deflections in the modern adaptive/smart structures. For such cases, a theoretical calculation based on linear or small deflection theory is likely to yield inaccurate results particularly for slender SMA beams at high intensity loads. The large deflection theory, on the other hand, invariably involves non-linear equations having no closed form solutions. That in turn leads to the different numerical techniques for solving those equations.

Study on superelastic SMA beam has been limited to small deflection theory [8]. Though large deflection analysis on slender superelastic SMA column has been carried out by Rahman [1], exclusive study on large deflection analysis on superelastic SMA beam is not reported in the literature. Observing these facts, a computer code based on 'C' has been developed using the finite difference technique (to solve boundary value problems) and also the Runge-Kutta technique (to solve initial value problems), for accurately analyzing the behaviors of the superelastic SMA beams. The load-deformation curves of the beams have been

predicted by using the developed code exploiting the nonlinear as well as the classical linear theories of the beams.

According to Fig. 1, for the case of small deflection (linear) theory, equations of the elastic curve of a beam are given as ,

$$\begin{aligned} \frac{1}{\rho} &= \frac{d\theta}{ds} \approx \frac{d\theta}{dx} \\ \Rightarrow \frac{1}{\rho} &= \frac{d^2y}{dx^2} = \frac{M}{EI} \\ \Rightarrow M &= EI \cdot \frac{d^2y}{dx^2} \dots\dots\dots(i) \end{aligned}$$

Where, ρ and M are the radius of curvature and the bending moment, respectively. The product EI (flexural rigidity) is usually constant along the beam length . Equation (i) is linear and applied for sufficiently small deflection of the beams. From differential calculus, however, for the exact value of $1/\rho$, the equation will

$$\text{become, } \frac{d^2y/dx^2}{[1 + (dy/dx)^2]^{3/2}} = \frac{M}{EI} \dots\dots(ii)$$

when dy/dx is very small, its square is negligible

compared to unity, and hence equation (i) becomes true. But as the value of dy/dx becomes larger it becomes necessary to imply equation (ii) for the determination of different design parameters particularly, the deflection and stresses of the beams.

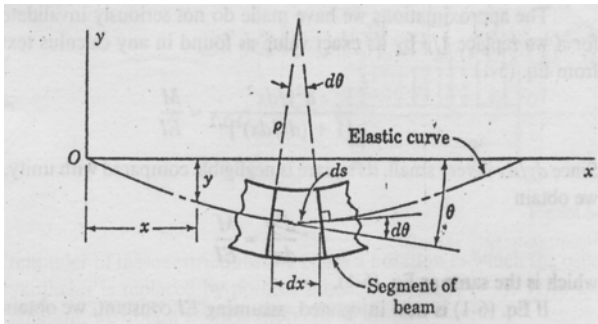


Fig.1 Parameters of the beam's geometry.

In general practice, analytical and exact solutions of beam deflection are desirable because of their ease of use and the insight they provide to designers. Specific geometric effects can be ascertained from these solutions. Especially for large deflections, however, numerical techniques such as – finite element analysis, boundary element analysis and finite difference analysis can be more accurate in predicting deflection behavior.

The work in this paper takes care of the effects of large deflections by numerical analysis. The analytical expression (based on small deflection theory) is compared to both numerical analysis results and experimental results for different end conditions of beams.

2. NUMERICAL SOLUTIONS

Solution were carried out, considering two different end conditions. Depending on the available boundary conditions they were solved either by Finite Difference Technique (for boundary value problem) or by Runge Kutta Method (for initial value problem). Experiments were performed to determine the Young's modulus (65Gpa) necessary for numerical analysis. Thus, nonlinearity considered in this study is purely geometrical.

Finite Difference Technique

It involves conversion of the governing differential equation as well as the boundary conditions in to a set of algebraic equations. If the governing equation is nonlinear the algebraic equations are also nonlinear. Finite Difference expressions of the nonlinear governing equation (ii) at any grid point i for the domain with step size h is given by:

$$EI \frac{y_{i+1} + 2y_i + y_{i-1}}{h^2} \left[1 + \left(\frac{4y_{i+1} - y_{i+2} + 3y_i}{h} \right)^2 \right]^{3/2} = M$$

The above 2nd order governing equation has an error of order h^2 . Evaluating the above expression at n number of grid points n algebraic nonlinear equations are obtained for the domain. Another two equations are obtained from the boundary conditions that must be compatible with the beam's deformation pattern. Those boundary conditions are also converted to algebraic equations through the finite difference expressions (forward difference at the initial boundary and backward difference at the final boundary so that order of error remains uniform, both for the governing equation and the boundary conditions). Thus there are $n+2$ unknowns and $n+2$ nonlinear algebraic equations for their determination. The set of nonlinear equations are solved by Newton-Raphson method.

It should be noted that for the linear theory only denominator becomes unity for the same expressions of governing equation. The boundary conditions remain the same. Numerical solutions of the linear theory should match closely with the exact solutions. It should be noted that for the iteration of the Newton-Raphson method to solve the nonlinear equations, initial guess values are the solutions of the linear equations. Thus, linear beam's equations (i) are solved first.

Integration Method to Solve Initial Value Problem

In this method, first of all, the 2nd order governing equation is converted to two first order equations.

Let $y=y_1$, $dy/dx=y_2$.

Therefore, the governing equation becomes,

$$\begin{aligned} dy_1/dx &= y_2 \\ dy_2/dx &= M/EI (1 + y_2^2)^{3/2} \end{aligned}$$

The two equations are then solved by initial value integration using Runge-Kutta method (error of order h^4). Therefore, this technique would yield much more accurate results than those from the finite difference technique.

Let's consider the two utmost end conditions (a) The Fixed-Free ends (Cantilever Beam) and (b) Both ends fixed beams.

Analysis of the Cantilever Beam

(Loaded by a point load P at the Tip)

Linear theory (second order Euler-Bernoulli equation) gives, for the elastic curve,

$$M(x) = EI \frac{d^2 y}{dx^2} - PL + Px = 0$$

Using the two necessary boundary conditions,

$$\text{At } x=0, y=0 \text{ and } \frac{dy}{dx} = 0$$

$$\Rightarrow EIy = -P \left(\frac{Lx^2}{2} - \frac{x^3}{6} \right)$$

Thus, at the tip, \Rightarrow

$$EIy = -PL^3 \left(\frac{1}{2} - \frac{1}{6} \right) = -\frac{PL^3}{6} (3-1) = -\frac{PL^3}{3}$$

The equation for large deflection (nonlinear) analysis is as follows

$$EI \frac{d^2 y}{dx^2} \left[1 + \left(\frac{dy}{dx} \right)^2 \right]^{3/2} - PL + Px = 0$$

Though the boundary conditions remain the same, the governing equation itself has become much more complicated than that obtained from the linear theory. Therefore, numerical techniques should be used to get the deflection. As, both the boundary values are given at the initial point, the solution has been obtained through initial value integration by Runge Kutta Method.

Analysis of the both ends fixed beam (Loaded at the mid-span by a point load P)

Through large deflection analysis:

$$EI \frac{d^2 y}{dx^2} \left[1 + \left(\frac{dy}{dx} \right)^2 \right]^{3/2} = \frac{P}{8} (4x - l)$$

Since the governing equation is of 2nd order, the used boundary conditions are:

$$\text{At } x_0 = 0 ; y = 0 \text{ and}$$

$$\text{At } x_i = l/2 ; \frac{dy}{dx} = 0$$

Therefore, it becomes a nonlinear boundary value problem having no closed form solutions. Thus, finite difference method has been used to get the accurate value of deflection and stress of the beams.

From the linear theory, however, the exact solution for deflection at any point x up to the mid span:

$$EIy = \frac{Fx^2}{48} (4x - 3l)$$

3. EXPERIMENTAL SETUP

A sufficiently rigid structure made of wood was constructed for conducting the experiment with different end conditions. Dead weight was applied at a point near the tip of the specimen (200mm long straight SMA rod with a diameter of 2mm) and the corresponding deflection was measured by the height gage (accuracy 0.01mm). In this case, there is a possibility of errors in reading, if the contact of height gage pointer with the

deflected specimen is carried out observing with bare human eyes. To eliminate such errors, a simple circuit with a DC source and a diode was wired between the specimen and the height gauge pointer. Therefore, whenever the height gage touches the beam it is indicated by the flash. The schematic of the process is shown in Fig. 2.

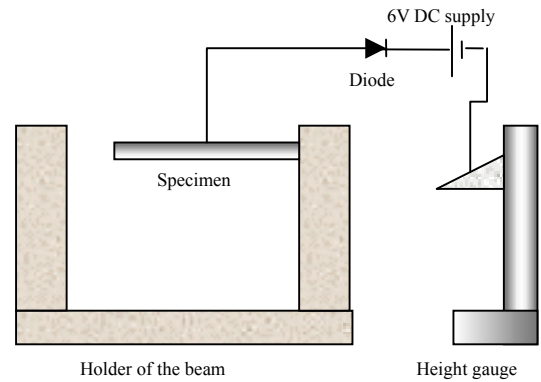


Fig.2 Experimental setup.

4. RESULTS AND DISCUSSIONS

In order to point out the deficiencies of the linear theories, the results of both the linear and nonlinear theories are shown in the same figures. Comprehensive results are obtained from experiments and compared with the analysis of cantilever SMA beam. As mentioned, superelastic beam material (NiTi) can recover too large strain (up to 7%). Moreover, stress-strain curve remain almost linear up to 1% strain [1-6]. Since the basic theme in this study is geometric nonlinearity of the beam (but linearly elastic beam material), the maximum strains corresponding to the numerical solutions are always within 1%.

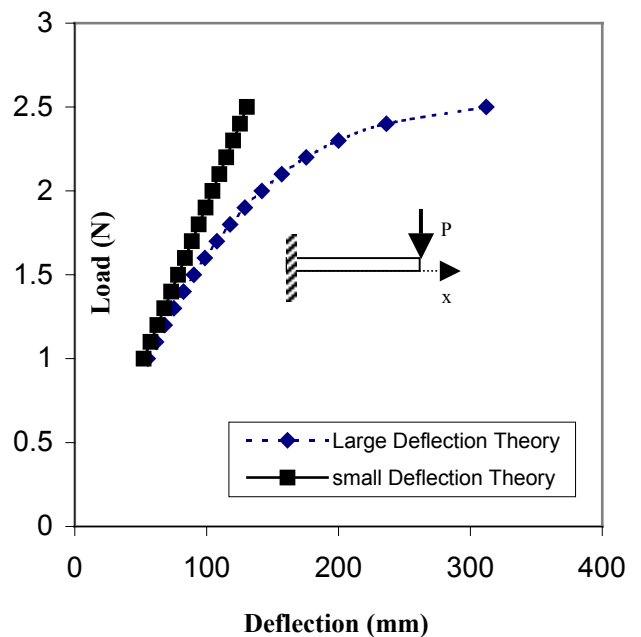


Fig. 3: Load-lateral deflection curve up to a strain limit of 1%.

Fig.3 clearly depicts the fact that the linear theory predicts lower values of the deflections in comparison to that of the nonlinear theory. While, from Fig.4, it is evident that the experimental result closely follows the large deflection theorem. So, it can be concluded that it is vitally important to consider nonlinear theory (large deflection theory) while designing slender cantilever beams particularly at high intensity load.

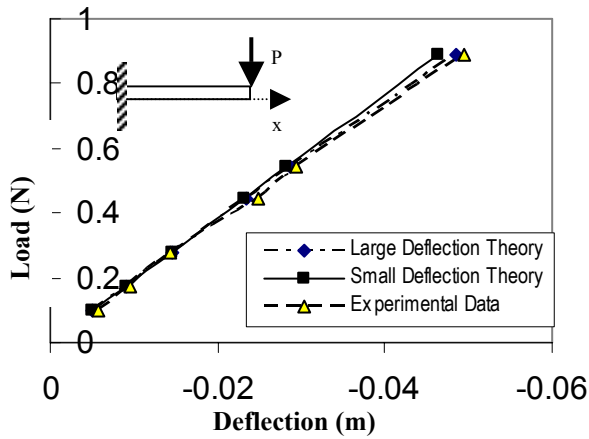


Fig.4 Comparison of numerical analysis and experimental results for an SMA beam [for $x=195\text{mm}$].

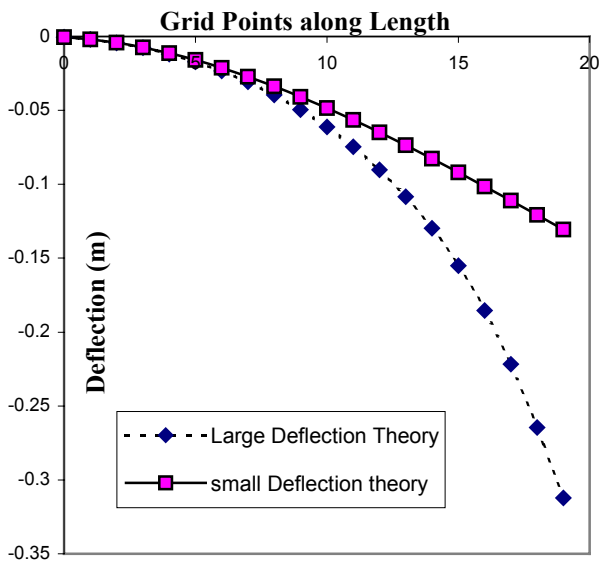


Fig.5 Deflection along the length of the specimen of the cantilever beam at a load of 2.5 N.

From Fig.5, again it is observed that small deflection (linear) theory fails to predict the correct deflections of the beam and consequently the solutions predicted by the nonlinear theory are much higher particularly at the grid points near the tip. Though not presented here, it is obvious that the large deflection analysis would show greater stress compared to the small deflection analysis, this again visualizes the importance of large deflection considerations.

It should be noted that shortening of the moment arm of

the cantilever beam is an important topic but it is not included in this paper for the sake of brevity.

For fixed ends beams only numerical results are presented in the following Figure that are obtained from the finite difference technique as described in the earlier section of this paper.

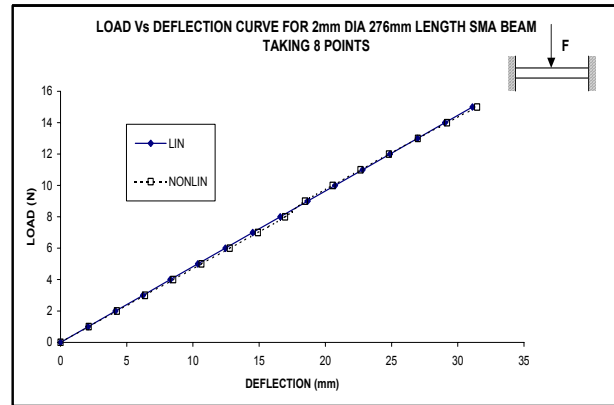


Fig.6 Comparison of linear and nonlinear solutions for both ends fixed superelastic SMA beams.

Because of fixed ends the rigidity is high and thus the deflection of the mid-span of a both ends fixed beam is too small compared to tip deflection of the cantilever beam. Therefore, the discrepancy between the linear and the nonlinear theories is not that much for the range of loading presented in Fig. 6. However, more accurate numerical methods to solve boundary value problem like Multisegment method of integration [9] is likely to yield better results for the case of large deflection analysis.

5. CONCLUSIONS

As the linear theory fails to account for the pronounced change in curvature and consequently predicts unrealistic solutions of elastic deflection and thus the stresses in the superelastic SMA beams, nonlinear theory of beam deflection is essential. Solutions are, thus, obtained for beams of different end conditions, using both linear and nonlinear theories in the present analysis, so that the shortcoming of linear theories in case of beams are verified and noted.

For large deflection analysis, geometric nonlinearity has been considered to investigate the elastic deflection of the slender superelastic SMA beams with two different end conditions. Results verify the fact that the linear or small deflection theory fails to predict the correct solutions particularly at high intensity loading. Discrepancy between the solutions predicted by the nonlinear and linear theories is more prominent for the cantilever beam than it is for the both ends fixed beams. Experimental results match closely with the nonlinear solutions that in turn, justifies the necessity of the present research.

6. REFERENCES

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