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A SINGLE PLATE THERMOACOUSTIC SYSTEM: VISCOUS CONJUGATE PROBLEM

Shohel Mahmud and Roydon Andrew Fraser

Department of Mechanical Engineering, University of Waterloo, ON., Canada.

ABSTRACT
A low Mach number oscillatory compressible flow of a viscous fluid near a single-plate thermoacoustic system are calculated analytically by simplifying the unsteady-compressible form of the Navier-Stokes equation and energy equation. The thickness of the plate is modeled in this paper and the problem is treated as a conjugate heat transfer problem. Analytical expressions for the velocity component, temperature inside the fluid and solid plate are derived based on a first order perturbation expansion. Finally, a wave equation is derived which is able to describe the characteristics of fluctuating pressure near the stack.

Keywords: Conjugate problem, Stack, Thermoacoustics, Viscous effect.

1. INTRODUCTION
Sound waves are ordinarily viewed as consisting of coupled pressure and displacement oscillations propagating through a gas. However, temperature oscillations always accompany the pressure changes [1]. The combination also of these three oscillations, and their interaction with solid boundaries produce a rich variety of 'thermoacoustic' effects. A thermoacoustic engine, whether it is a heat pump or prime mover, consists of four basic parts; specifically (a) a speaker, (b) a resonant tube, (c) a heat exchanger, and (d) a stack. The stack serves as the 'heart' of any thermoacoustic device and plays a key role in the operation of such devices. Positioned between the hot and cold heat exchangers, the stack determines how much and in what direction heat and work transfer will occur. The stack is the physical structure that imposes boundary conditions on the oscillating gas, and provides the critical phase relationships necessary for beneficial heat or work transfer. Starting from the single plate, stacks are available in different sizes and shapes. Multi-plate arrays, honeycombs, spiral roles, and pin arrays' are some example of stacks commonly used in thermoacoustic engines and refrigerators [2]. Stacks made of parallel plates are simple in construction and are popular stack geometry for thermoacoustic devices. A single plate placed in front of an acoustic standing wave is the simplified version of thermoacoustic stack used to describe many fundamental thermoacoustic phenomena.

For a single plate, Swift [3], Raspet *et al.* [4], Santillan and Boullosa [5], Wetzel and Herman [6], and Wetzel and Herman [7] performed analytical and/or experimental works on single plate thermoacoustic system. All of the above works considered the standing wave features for fluctuating pressure and fluctuating velocity even though these fluctuating features are modified by the presence of the plate itself. None of them considered the finite thickness of the plate.
In this paper, we focus on the hydrodynamic and thermal behavior of an oscillating fluid around a single plate stack of a multi-stacks thermoacoustic device. Analytical expressions for velocity, temperature are derived after simplifying and solving the momentum and energy equations.

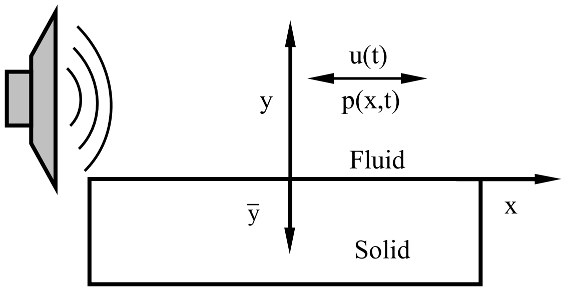

Fig. 1 Schematic diagram of the problem

2. PROBLEM FORMULATION
Figure 1 shows a single plate stack exposed to an oscillating compressible gas. First to be derived is an expression for axial velocity as a function of transverse direction. Considering the momentum equation for a compressible viscous flow [8]

$$\rho\left[\frac{\partial \mathbf{V}}{\partial t}+\left(\mathbf{V}.\vec{\nabla}\right)\mathbf{V}\right] = -\vec{\nabla}p+\mu\vec{\nabla}^2\mathbf{V}+\left(\xi+\frac{\mu}{3}\right)\vec{\nabla}\left(\vec{\nabla}.\mathbf{V}\right) \quad (1)$$

where μ and ξ are dynamic and second viscosity, respectively. To simplify Eq. (1) the following was argued: it is assumed that $u_1/v_1 \geq \tilde{\lambda}/\delta_v$; $\partial/\partial x$ is of order $1/\tilde{\lambda}$ and $\partial/\partial y$ is order of $1/\delta_v$; and $\delta_v << \tilde{\lambda}$. v_1 is the transverse velocity component, $\tilde{\lambda}$ is the radian length of the sound wave, $\delta_v = \sqrt{2\nu/\omega}$ is the viscous penetration depth. Equation (1) then reduces to

$$\frac{\partial^2 \hat{u}_1}{\partial y^2}-\left(\sqrt{\frac{i\omega}{\nu}}\right)^2\hat{u}_1 = -\frac{1}{\mu}\frac{\partial \hat{p}_1}{\partial x} \quad (2)$$

Using $\hat{\varphi}=\varphi e^{i\omega t}$ (where φ is any variable, for example, u, s, p etc.), the general solution to the momentum equation (Eq. (2)) is

$$u_1 = C_1 e^{\left(\frac{1+i}{\delta_v}y\right)}+C_2 e^{-\left(\frac{1+i}{\delta_v}y\right)}-\frac{i}{\omega\rho_m}\frac{\partial p_1}{\partial x} \quad (3)$$

subjected to the boundary conditions, $u_1(0)=0$ and $u_1(\infty)$=finite. The final expression for u_1 becomes

$$u_1 = \frac{i}{\omega\rho_m}\frac{\partial p_1}{\partial x}\left\{1-\exp\left(-\frac{1+i}{\delta_v}y\right)\right\} \quad (4)$$

The energy equation, according to Landau and Lifshitz [8], for the fluid is

$$\rho_f C_p\left[\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}\right] = k_f\left[\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right]+\beta T\left[\frac{\partial p}{\partial t}+u\frac{\partial p}{\partial x}+v\frac{\partial p}{\partial y}\right] \quad (5)$$

where β, k_f, ρ_f, and C_p are thermal expansion coefficient, fluid thermal conductivity, fluid density, and specific heat of the fluid at constant pressure, respectively. Linearising Eq. (5) after introducing a perturbation expansion and keeping only the first order terms, Eq. (5) becomes after some rearrangement

$$\frac{\partial^2 T_1}{\partial y^2}-\left(\sqrt{\frac{i\omega C_p\rho_m}{k_f}}\right)^2 T_1 = \frac{C_p\rho_m}{k_f}\frac{\partial T_m}{\partial x}u_1-\frac{i\omega\beta T_m p_1}{k_f} \quad (6)$$

and the general solution to Eq. (6) is

$$T_1 = C_1 e^{\left(\frac{1+i}{\delta_k}y\right)}+C_2 e^{-\left(\frac{1+i}{\delta_k}y\right)}+\frac{\Pr}{\Pr-1}\frac{\nabla T_m}{\omega^2\rho_m}\times\frac{\partial p_1}{\partial x}e^{-\left(\frac{1+i}{\delta_v}y\right)}-\left[\frac{\nabla T_m}{\omega^2\rho_m}\frac{\partial p_1}{\partial x}-\frac{\beta T_m p_1}{\rho_m C_p}\right] \quad (7)$$

The boundary conditions for Eq. (7) are $T_1(0)=T_w$ and $T_1(\infty)$=finite. One needs to solve the energy equation for the solid region prior to calculating the unknown constants of Eq. (7). The energy equation for the solid region is

$$\rho C_s\frac{\partial T_s}{\partial t} = k_s\frac{\partial^2 T_s}{\partial \bar{y}^2} \quad (8)$$

where ρ, C_s, and k_s are solid density, specific heat of the solid, and solid thermal conductivity. Linearising and keeping the first order term only, Eq. (8) becomes

$$i\omega\rho_s C_s T_{s1} = k_s\frac{\partial^2 T_{s1}}{\partial \bar{y}^2} \quad (9)$$

The general solution to Eq. (9) is

$$T_{s1} = C_1 e^{\sqrt{\frac{i\omega\rho_s C_s}{k_s}}\bar{y}}+C_2 e^{-\sqrt{\frac{i\omega\rho_s C_s}{k_s}}\bar{y}} \quad (10)$$

subjected to the boundary conditions $T_{s1}(0)=T_w$ and $T_{s1}(\infty)$=finite. This gives the final expression for the temperature distribution inside the solid region as

$$T_{s1} = T_w e^{-\sqrt{\frac{i\omega\rho_s C_s}{k_s}}\bar{y}} \quad (11)$$

Furthermore, the solid-fluid interface satisfies the boundary condition

$$k_f\left.\frac{\partial T_1}{\partial y}\right|_{y=0} = -k_s\left.\frac{\partial T_{s1}}{\partial \bar{y}}\right|_{\bar{y}=0} \quad (12)$$

Using Eqs. (7), (11), and (12), and after long calculations one can obtain

$$T_1 = \frac{\beta T_m p_1}{\rho_m C_p}-\left\{\frac{\beta T_m p_1}{\rho_m C_p}+\frac{1+\varepsilon_s\sqrt{\Pr}}{\Pr-1}\frac{\nabla T_m}{\omega^2\rho_m}\frac{\partial p_1}{\partial x}\right\}\times\frac{e^{-\left(\frac{1+i}{\delta_k}y\right)}}{1+\varepsilon_s}-\frac{\nabla T_m}{\omega^2\rho_m}\frac{\partial p_1}{\partial x}\left\{1-\frac{\Pr}{\Pr-1}e^{-\left(\frac{1+i}{\delta_v}y\right)}\right\} \quad (13)$$

where Pr is the Prandtl number of the fluid. The parameter ε_s can be defined by

$$\varepsilon_s = \sqrt{\frac{\rho_m C_p k_f}{\rho_s C_s k_s}} \quad (14)$$

which takes care the influence of solid properties on the fluid temperature and vice versa. It should be noted that the expressions of velocity and temperature of fluid have a term p_1. The term p_1 is the first order pressure which needs to be determined using a wave equation.

3. WAVE EQUATION AND SOLUTION

The linearized first order continuity and inviscid momentum equations are given in Eqs. (15) and (16) after a perturbation expansion

$$i\omega\rho_1 + \frac{\partial(\rho_m u_1)}{\partial x} = 0 \quad (15)$$

$$i\omega\rho_m u_1 = -\frac{\partial p_1}{\partial x} \quad (16)$$

Differentiating Eq. (16) and combining with Eq. (15), one obtains

$$\rho_1 = -\frac{1}{\omega^2}\frac{\partial^2 p_1}{\partial x^2} \quad (17)$$

Using the following thermodynamic relation

$$\rho_1 = -\rho_m \beta T_1 + (\gamma / c^2) p_1 \quad (18)$$

ρ_1 can be eliminated from Eq. (17) to get

$$\frac{\partial^2 p_1}{\partial x^2} + \left(\frac{\omega\sqrt{\gamma}}{c}\right)^2 p_1 - \rho_m \beta \omega^2 T_1 = 0 \quad (19)$$

which can be reduced into Eq. (20) by using the expression of temperature (Eq. (13)).

$$\frac{\partial^2 p_1}{\partial x^2} + \beta \nabla T_m \frac{\partial p_1}{\partial x} + \left[\left(\frac{\omega\sqrt{\gamma}}{c}\right)^2 - \frac{T_m \beta^2 \omega^2}{C_p}\right] p_1 = 0 \quad (20)$$

Using the speed of sound and heat capacity thermodynamic relations, $c^2=\gamma RT_m$ and $C_p=\gamma R/(\gamma-1)$, the terms inside the square bracket of Eq. (20) simplify to become $(\omega/c)^2$. Finally, the pressure equation for this problem becomes

$$\frac{\partial^2 p_1}{\partial x^2} + \frac{\partial \ln(T_m)}{\partial x}\frac{\partial p_1}{\partial x} + \left(\frac{\omega}{c}\right)^2 p_1 = 0 \quad (21)$$

The general solution to the wave equation described in Eq. (21) is

$$p_1 = C_1 e^{\psi_1 x} + C_2 e^{\psi_2 x} \quad (22)$$

where the constants ψ_1 and ψ_2 are

$$\psi_1 = \frac{-\partial \ln T_m}{2\partial x} - \frac{1}{2}\sqrt{\left(\frac{\partial \ln T_m}{\partial x}\right)^2 - \left(\frac{2\omega}{c}\right)^2}$$
$$\psi_2 = \frac{-\partial \ln T_m}{2\partial x} + \frac{1}{2}\sqrt{\left(\frac{\partial \ln T_m}{\partial x}\right)^2 - \left(\frac{2\omega}{c}\right)^2} \quad (23)$$

At the starting point of the plate ($x=x_s$), it is appropriate to apply the standing wave pressure $p^s(x_s)$. However, it is extremely difficult to apply an appropriate boundary condition at the plate exit ($x=x_e$). None of the existing thermoacoustic literatures gives an idea about such boundary condition. In such situation, the more logical way to set the exit boundary condition similar to the fluctuating feature of a standing wave, i.e. $p^s(x_e)$. Then the constants, C_1 and C_2 of Eq. (22), can be calculated in the following forms

$$C_1 = \frac{p^s(x_s)e^{\psi_2 x_e} - p^s(x_e)e^{\psi_2 x_s}}{e^{\psi_1 x_s}e^{\psi_2 x_e} - e^{\psi_1 x_e}e^{\psi_2 x_s}}$$
$$C_2 = -\frac{p^s(x_s)e^{\psi_1 x_e} - p^s(x_e)e^{\psi_1 x_s}}{e^{\psi_1 x_s}e^{\psi_2 x_e} - e^{\psi_1 x_e}e^{\psi_2 x_s}} \quad (24)$$

where the definitions of different pressures in Eq. (24) are given by

$$p^s(x) = P_A \sin\left(x/\tilde{\lambda}\right)$$
$$p^s(x_s) = P_A \sin\left(x_s/\tilde{\lambda}\right)$$
$$p^s(x_e) = P_A \sin\left(x_e/\tilde{\lambda}\right) \quad (25)$$

where P_A is the pressure amplitude determined by the drive ratio ($=P_A/P_{\text{atm.}}$) for a particular problem. Finally, the expression of p_1 becomes

$$p_1 = \left[\frac{p^s(x_s)e^{\psi_2 x_e} - p^s(x_e)e^{\psi_2 x_s}}{e^{\psi_1 x_s}e^{\psi_2 x_e} - e^{\psi_1 x_e}e^{\psi_2 x_s}}\right]e^{\psi_1 x} - \left[\frac{p^s(x_s)e^{\psi_1 x_e} - p^s(x_e)e^{\psi_1 x_s}}{e^{\psi_1 x_s}e^{\psi_2 x_e} - e^{\psi_1 x_e}e^{\psi_2 x_s}}\right]e^{\psi_2 x} \quad (26)$$

The applicability of Eq. (26) is restricted by the relations given in Eq. (23). To get a real result from Eq. (26) the following condition must be satisfied

$$\frac{\partial \ln T_m}{\partial x} = \frac{\nabla T_m}{T_m} \geq \frac{2\omega}{c} \quad (27)$$

which, after further mathematical operations and simplifications, becomes

$$\nabla T_m \geq 4\pi \frac{T_m}{\lambda} = \frac{2T_m}{\tilde{\lambda}} \quad (28)$$

4. CONCLUSION

We modeled the stack of a thermoacoustic device as a single plate exposed in front of an oscillating compressible gas. Fluid is assumed viscous-compressible. The governing equations for velocity, temperatures inside the fluid and solid are simplified using a first order linear perturbation method. Expressions for velocity, temperatures in solid and fluid are derived. Each expression shows a real and an imaginary part. Our main goal is the real part of each expression. Determining the expression of pressure in terms of the standing wave characteristics completes the analysis.

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