

DETERMINATION OF STRESS CONCENTRATION FACTORS IN RECTANGULAR STEPPED BAR UNDER DIFFERENT LOADING

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ABSTRACT

Analyzing the stress concentration in the critical region of a mixed boundary value elastic problem has always been a major area of interest in fracture mechanics. In most of the cases, the problems were circumvented with numerical solution. This study uses formulation of displacement potential function in finite difference method to determine the stress concentration factor in the critical fillet region of a stepped bar under tensile load and bending moment. For simplicity, the stepped bar is considered as a two-dimensional plane strain problem. The results obtained have been presented graphically and are in good agreement with the available result in the literature

Keywords: Stress analysis, Finite difference technique, Stress concentration factor.

1. INTRODUCTION

This paper is an attempt to determine the value of stress concentration factor in a stepped bar under the condition of uniform tensile load and bending moment. In this regard we have used displacement potential function formulation of two-dimensional elastic problems, which enables us to manage the mixed mode of the boundary conditions as well as the zones of their transition.

Now-a-days elasticity is a classical topic and its problems are even more classical. But the stress analysis problems are still suffering from a lot of shortcomings [1-5]. It has been often failed in establishing a very good correlation between theoretical analyses and experimental observation. To make-up this problem, it has been conjectured the behavior of materials in terms of its ultimate strength, yield strength, endurance strength, and fracture strength, but still could not really satisfactorily account for the shortcomings. Two factors may really be responsible for it. Both of these factors involve selection of the boundary of elastic problems: one is the boundary condition and the other is the boundary shape. The necessity of the management of boundary shape has lead to the invention of the finite element technique and its overwhelming popularity, especially because of the side by side development of high-powered computing machines. Of course, the adaptation of the finite-element method relieved us from our major inability of managing arbitrary boundary shapes but we are constantly aware of its lack of sophistication and doubtful quality of the solutions so obtained.

The other factor of impediment to quality solutions of

elastic problems is the treatment of the transition in boundary conditions. Elastic problems are either formulated in terms of deformation parameters or stress parameters. But, at boundary, all the problems are invariably the mixture of both known deformations and known stress boundary conditions. But neither of the two formulations would allow us to account fully both these two types of boundary conditions with equal precision and sophistication in the region of transition where boundary conditions are changing from one type to the other.

The formulation of two-dimensional elastic problems used here was first introduced by Uddin [6], later Idris *et. al.* [7-8] used it for obtaining analytical solutions of a number of mixed boundary-value elastic problems and Ahmed [9-12] extended its use where he obtained finite-difference solutions of a number mixed boundary value problems of simple rectangular bodies. Later, Akanda developed a new numerical scheme [13-15] by which he solved irregular shaped elastic bodies under mixed mode of loading. This study focuses the solution of the problems of rectangular stepped bar using the Akanda's developed numerical scheme. Solutions especially at the fillet region, which are observed as the most critical zone, are looked into. Analyzing the solution obtained, stress concentration factors are determined. Effects of fillet radius and the thickness ratio of the stepped bar on the stress concentration factors have also been studied. The rationality and reliability of solution is checked by comparing the results obtained with those available in the literature. The stress concentration factor (K_t) is presented graphically as a function of fillet ratio (r/h) and height ratio (H/h) for

body under bending moment as well as under uniform tensile load.

2. GOVERNING EQUATION IN TERMS OF DISPLACEMENT POTENTIAL FUNCTION

Analysis of stresses in a body is usually a three-dimensional problem. Fortunately, in the cases of plane stress or plane strain, the stress analysis of three-dimensional body can easily be treated as a two-dimensional one. In our case the problem of stepped bar is considered as a plane strain problem. In the case of the absence of any body forces, the equations governing the three stress components σ_x , σ_y and σ_{xy} under the states of plane stress or plane strain are:

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = 0 \quad (1)$$

$$\frac{\partial \sigma_y}{\partial y} + \frac{\partial \sigma_{xy}}{\partial x} = 0 \quad (2)$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) (\sigma_x + \sigma_y) = 0 \quad (3)$$

Replacement of the stress components in Eqs.(1-2) by their relations with the displacement components u and v makes Eq. (3) redundant and transforms Eqs. (1) and (2) to

$$\frac{\partial^2 u}{\partial x^2} + \left(\frac{1-\mu}{2} \right) \frac{\partial^2 u}{\partial y^2} + \left(\frac{1+\mu}{2} \right) \frac{\partial^2 v}{\partial x \partial y} = 0 \quad (4)$$

$$\frac{\partial^2 v}{\partial y^2} + \left(\frac{1-\mu}{2} \right) \frac{\partial^2 v}{\partial x^2} + \left(\frac{1+\mu}{2} \right) \frac{\partial^2 u}{\partial x \partial y} = 0 \quad (5)$$

The problem thus reduces to finding u and v in a two-dimensional field satisfying the two elliptic partial differential equations (4) and (5).

In this paper, the problem is reduced to the determination of a single function instead of two functions u and v , simultaneously, satisfying the equilibrium equation (4) and (5). In this formulation, as in the case of Airy's stress function $\phi(x,y)$, a potential function $\psi(x,y)$ [6] is defined in terms of displacement components as

$$u = \frac{\partial^2 \psi}{\partial x \partial y} \quad (6)$$

$$v = -\frac{1}{1+\mu} \left[(1-\mu) \frac{\partial^2 \psi}{\partial y^2} + 2 \frac{\partial^2 \psi}{\partial x^2} \right]$$

When the displacement components in Eqs. (4) and (5) are replaced by $\psi(x,y)$, Eq.(4) is automatically satisfied and the only condition that $\psi(x,y)$ has to satisfy becomes

$$\frac{\partial^4 \psi}{\partial x^4} + 2 \frac{\partial^4 \psi}{\partial x^2 \partial y^2} + \frac{\partial^4 \psi}{\partial y^4} = 0 \quad (7)$$

Therefore, the problem is now formulated in such a way that a single function $\psi(x, y)$ has to be evaluated from the bi-harmonic equation (7), satisfying the boundary conditions that are specified at the boundary.

3. BOUNDARY CONDITION WITH ψ -FORMULATION

The boundary condition are known in terms of the normal and tangential components of displacement u_n and u_t , and of stress σ_n and σ_t at any point on an arbitrary shaped boundary. These four components are expressed in terms of σ_x , σ_y , σ_{xy} , u , and v . The components of stress and displacement with respect to the reference axes x and y of the body, are expressed as follows:

$$u_n = ul + vm \quad (8)$$

$$u_t = vl - um \quad (9)$$

$$\sigma_n = \sigma_x l^2 + 2\sigma_{xy} lm + \sigma_y m^2 \quad (10)$$

$$\sigma_t = (l^2 - m^2) \sigma_{xy} + l.m(\sigma_y - \sigma_x) \quad (11)$$

Here l and m are the direction cosines of the normal to the boundary. The boundary conditions at any point on the boundary are specified in terms of any two values of u_n , u_t , σ_n and σ_t . In order to solve the mixed boundary-value problems of irregular-shaped bodies using the present formulation, the boundary conditions need to be expressed in terms of $\psi(x,y)$ and this can be done by substituting the following expressions of stress along with the displacement components in Eq. 6 with respect to reference axes x and y in Eqs. (8-11).

$$\sigma_x(x, y) = \frac{E}{(1+\mu)^2} \left[\frac{\partial^3 \psi}{\partial x^2 \partial y} - \mu \frac{\partial^3 \psi}{\partial y^3} \right] \quad (12)$$

$$\sigma_y(x, y) = -\frac{E}{(1+\mu)^2} \left[\frac{\partial^3 \psi}{\partial y^3} + (1+\mu) \frac{\partial^3 \psi}{\partial x^2 \partial y} \right] \quad (13)$$

$$\sigma_{xy}(x, y) = \frac{E}{(1+\mu)^2} \left[\mu \frac{\partial^3 \psi}{\partial x \partial y^2} - \frac{\partial^3 \psi}{\partial x^3} \right] \quad (14)$$

As far as numerical method of solution of bi-harmonic equation is concerned, it is evident from the expressions of boundary conditions (8-11) that ψ has to satisfy bi-harmonic equation within the body and any of Eqs. (8-11) at the points on the boundary.

4. ELASTIC PROBLEM AND ITS BOUNDARY CONDITIONS

The geometry of the selected problem and their boundary conditions are shown in Figs 1 and 2. Figure 1 describes the problem of stepped bar under loading as a cantilever. Figure 2 describes the same geometrical

problem considered under axial tension. The end surface of the wider end of the stepped bar is considered rigidly fixed, therefore, the boundary conditions are given as $u_n=0, u_t=0$. The other surfaces, those are free from loading, the boundary conditions are given as $\sigma_n=0, \sigma_t=0$. The dimensions width, H , and length, b , are kept fixed and varying the fillet radius, r , and smaller thickness, h , different problems are solved for different H/h and r/h values.

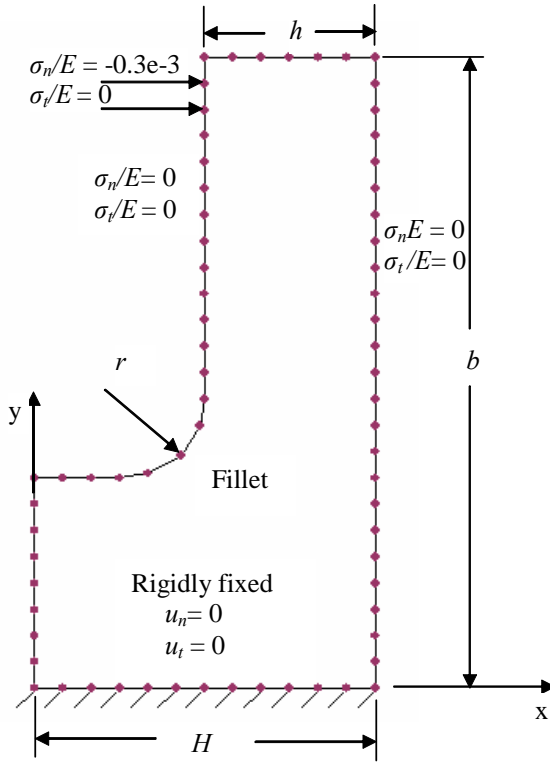


Fig 1. Elastic problem with boundary conditions for bending moment (profile $H/h = 2.00$; $r/h = 1.00$).

5. NUMERICAL SOLUTION

In practice the whole region of the elastic problem is divided into a desire number of mesh points and the value of dependent function are evaluated only at these points. Considering mixed boundary value problem, boundary conditions may be specified into four combinations: (u_n, u_t) ; (σ_n, σ_t) ; (u_n, σ_t) and (σ_n, u_t) . It had been seen that the boundary conditions are given in terms of derivatives of the function ψ . These differential equations have been expressed in form of difference equations. As two boundary conditions are specified on the physical boundary, the finite difference expressions of the differential equations associated with the boundary conditions are applied to the same point on the boundary. So, two linear algebraic equations are assigned to a single point on the boundary. The computer program [15] is organized in such a way that out of two equations, one is applied to the physical boundary grid point and the other is applied to a point exterior to the physical boundary. The discretized form of bi-harmonic equation in terms of

ψ applied to any interior point will give rise a single algebraic equation. Therefore, a system of the algebraic equations has been solved by lower-upper decomposition method.

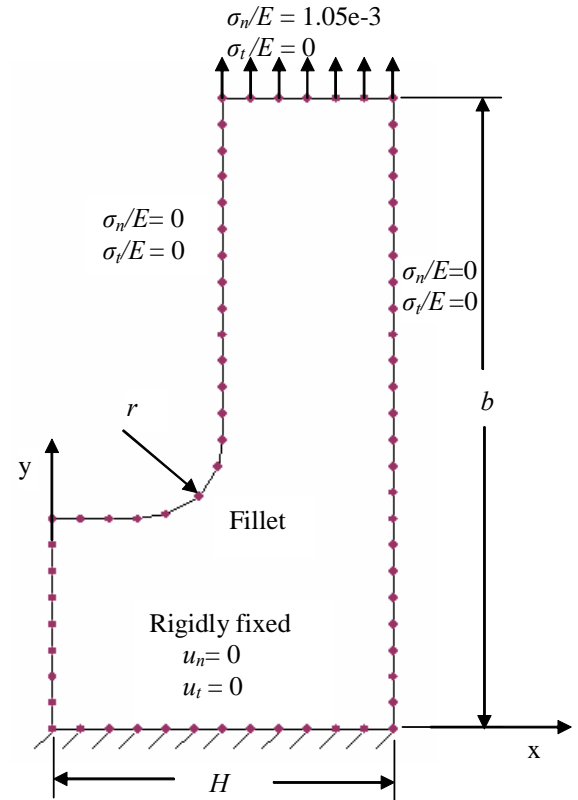


Fig 2. Elastic problem with boundary conditions for uniform tensile load (profile $H/h = 2.00$; $r/h = 1.00$).

6. RESULTS AND DISCUSSION

For the body under uniform tension the variation of stress concentration factor (K_t) as a function of fillet radius and thickness is presented in Fig 3. It is observed

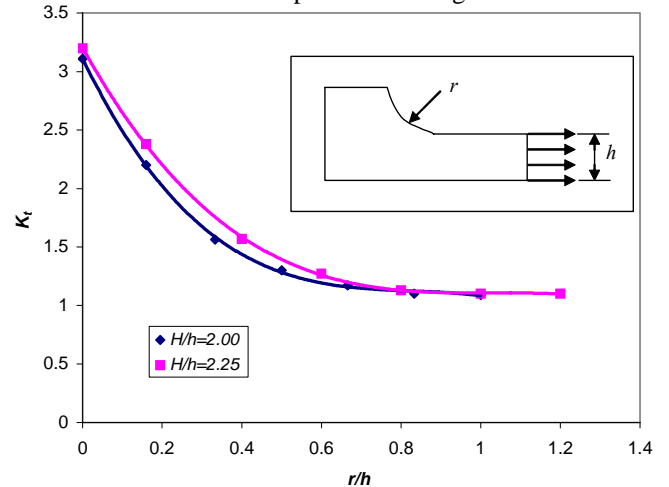


Fig 3. Stress concentration factor (K_t) vs. r/h plot under uniform tensile load.

that the values of K_t obtained for $H/h = 2.25$ and $H/h = 2.00$ are almost identical throughout r/h ratios. So for stepped bar under uniform tensile loading, the thickness ratios has no significant effect on the stress concentration. Value of K_t rises with decreasing fillet radius. At $r/h=0$ complete sharp corner is produced. Maximum stress concentration found at that point is about 3.2. For $r/h > 0.6$, the stress concentration factor is insignificant and found very close to unity.

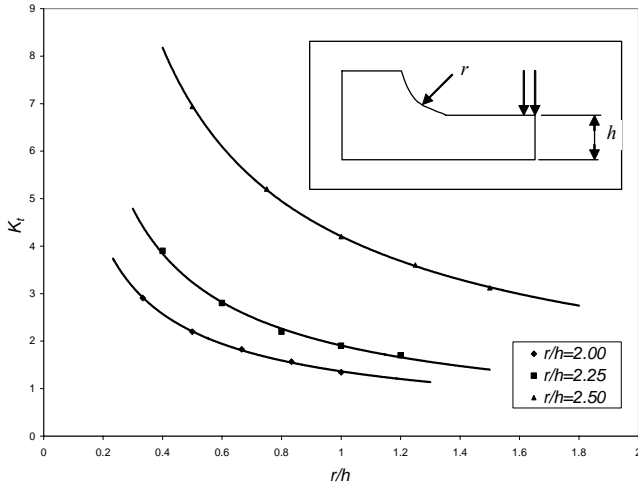


Fig 4. Stress concentration factor (K_t) vs. r/h plot under bending moment.

The variation of stress concentration factor (K_t) for the stepped cantilever has been presented in Fig 4. as a function of fillet radius (r/h) and thickness ratio (H/h). Critical section for calculation of stress concentration factor has been observed at the root of smaller dimension of the rectangular bar. From the figure it is observed that the stress concentration factor increases with the increase in thickness ratio (H/h) and decreases with the increase in fillet radius. For a fillet ratio, $r/h = 0.5$ and for thickness ratios, ($H/h=$) 2.0 and 2.5 the stress concentration factors are 2.2 and 6.96 respectively. So, the thickness ratios play a vital effect on the stress concentration factor for body under bending. From the trend it is also clear that the variation of stress concentration factor is very high for $r/h < 0.4$. At $r/h = 0$ sharp corner is produced at the root of smaller dimension. Maximum stress concentration is found at that point but it is not presented here.

Figure 5 shows the comparison of K_t found from the present study with that found in the experiment by Weibel [5]. It is noteworthy that in Weibel study the stepped bar was doubly filleted. The comparison shows that up to $r/h = 0.5$ two results are almost identical. Some discrimination has been found if Weibel's curve is extended further for $r/h > 0.5$. In the Weibel's result it is also seen that for $H/h = 1.5$ and $H/h = 3.00$ there is almost no difference on values of K_t which fact is also in agreement with the present study.

In Fig 6 comparison of stress concentration is also made with the results of brittle material obtained by Anita Noble [16]. In our study we used the material whose poisson's ratio is 0.3. Anita claims that the

maximum stress concentration factor results in a value less than that found for the theoretical value. Since brittle materials cannot plastically deform, the stress raisers will create the theoretical stress concentration situation.

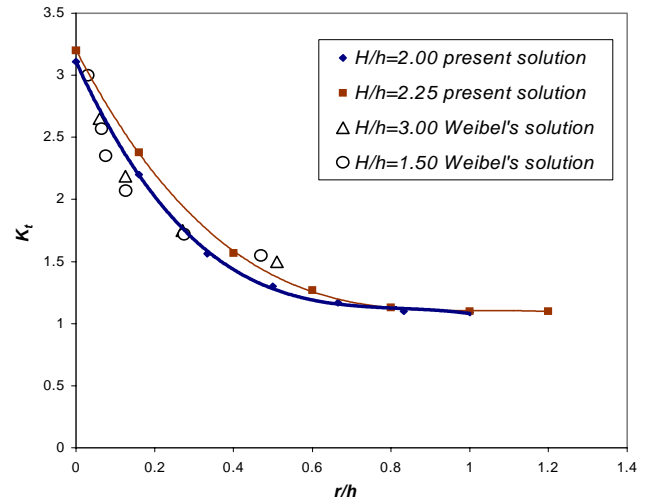


Fig 5. Comparison of our result with Weibel's result for K_t .

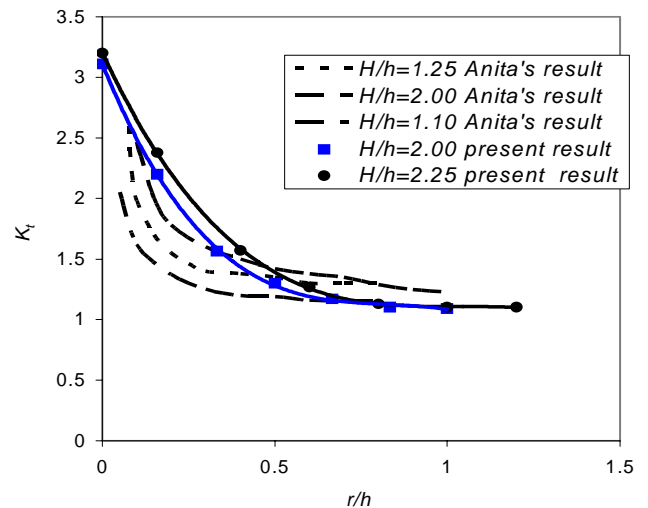


Fig 6. Comparison the result with result obtained by Anita Noble.

For between $r/h > 0.4$ this two solutions are almost identical but for lower value of r/h , the value of K_t found by the present study is higher than Anita's results for K_t . This difference is due to the difference in material and geometry.

7. CONCLUSIONS

From the figures it is observed that stress concentration varies with the fillet radius in such a manner that highest stress developed at zero fillet radius as sharp corner is produced. There is very low effect on stress concentration in higher radius ratios. The philosophy of the present program using formulation of displacement potential function ψ is such that it encompasses all sort of practical considerations, including sharp discontinuities and mixed mode

boundary conditions. So, procedure of the solution for stress concentration factor obtained in this paper in the rectangular stepped bar can be applied to determine K_t in any kind of arbitrary shaped object. As superiority of the present solution scheme over the existing approaches is its ability in satisfying the boundary conditions exactly, the solutions for K_t obtained by the present program are promising and satisfactory for the entire critical region of interest. Therefore, this study eliminates the use of experimental techniques especially the photo-elastic method.

8. REFERENCES

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9. NOMENCLATURE

Symbol	Meaning	Unit
k	Mesh length in x and y direction	(m)
r	Fillet radius	(m)
E	Modulus of elasticity	(Pa)
$\psi(x,y)$	Displacement potential function	-
μ	Poisson's ratio	-
u	Displacement component in x -direction	(m)
v	Displacement component in y -direction	(m)
σ_x	Stress component in x -direction	(Pa)
σ_y	Stress component in y -direction	(Pa)
u_n	Displacement component normal to the boundary	(m)
u_t	Displacement component tangential to the boundary	(m)
σ_n	Stress component normal to the boundary	(Pa)
σ_t	Stress component tangential to the boundary	(Pa)
σ_{xy}	Shear Stress component in xy -plane	(Pa)
$\phi(x,y)$	Airy's stress function	-
H	Larger dimension of stepped bar	(m)
h	Smaller dimension of stepped bar	(m)
b	Length of stepped bar	(m)
K_t	Stress concentration factor	-