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A NEW FORMULATION FOR SOLUTION OF ELASTIC PROBLEM BY FINITE DIFFERENCE TECHNIQUE

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ABSTRACT
This paper presents a new formulation for the solution of elastic problem based on a finite difference technique. The displacement potential function formulation is well known to solve this type of problem in finite difference technique. But the problem is that the formulation is asymmetric both in stress and displacement components. For this reason the discretized formula structure of boundary conditions was such that some formulas are not applicable in some specific points on boundary of the rectangular discretized field. So the complexity of computer programming increases and some times the coefficient matrix makes an ill condition. In this paper, a new formulation is proposed where the structures of the formula of boundary conditions are symmetric with respect to *x* and *y* axes. A computer program is developed using FORTRAN language, which solves plane stress and plane strain problems of elastic bodies with mixed boundary conditions.

Keywords: Finite difference technique, Symmetric formula, Stress analysis, Deep cantilever.

1. INTRODUCTION
This paper is an attempt to overcome the difficulties faced in the management of boundary conditions. It uses a new formulation of two-dimensional elastic problems, which enables us to manage the mixed mode in the boundary conditions as well as the zones of their transitions. The computational work in this formulation is of the same magnitude as in the Uddin's displacement potential function formulation, in case of numerical approach of solutions.

Solution of two dimensional elastic problems by finite difference technique was first succeeded by Uddin's formulation in Ref.[1], Idris used it in Ref.[2-3] for obtaining analytical solutions of a number of mixed boundary-value elastic problems, and Ahmed extended its use in Ref.[4-7] where he obtained finite-difference solutions of a number of mixed boundary value problems of simple boundary shapes. Later Akanda *et. al*. [8-10] extends the earlier works to include the problems of arbitrary boundary shapes.

Uddin's formulation had some basic limitations that we tried to resolve in a systematic and most efficient way. The basic limitation that was faced in Uddin's formulation was like the formulae for stress and displacement components are asymmetric in *x* and *y* coordinates. Thus in finite difference technique the structure of some formula could not be used in outer boundary of the rectangular meshed elastic body. Therefore, the computer code to solve the problem becomes much complex, lengthy and also loses user-friendship.

In this paper a new formulation is proposed. I this formulation the structures of different boundary value formulae are obtained in symmetric form. The developed structure can be used in both inner and outer boundary and also in any portion of the boundary. We have used 'FORTRAN' language to develop a computer code. The code is very simple, short in length than before and also easy to understand for a new user. To justify the proposed formulation a simple elastic problem has been solved and the numerical results are analyzed.

2. GOVERNING EQUATION

2.1. Governing Equation in Terms of Displacement Potential Function, *ψ*
In the case of the absence of any body forces, the equations governing the three stress components $\sigma_x, \sigma_y, \sigma_{xy}$ under the states of plane stress or plane strain are:

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = 0 \tag{1}$$

$$\frac{\partial \sigma_y}{\partial y} + \frac{\partial \sigma_{xy}}{\partial x} = 0 \tag{2}$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(\sigma_x + \sigma_y\right) = 0 \tag{3}$$

Replacement of the stress components in equations.

(1) to (3) by their relations with the displacement components u and v makes equation (3) redundant and equations (1) and (2) transforms to

$$\frac{\partial^2 u}{\partial x^2}+\left(\frac{1-\mu}{2}\right)\frac{\partial^2 u}{\partial y^2}+\left(\frac{1+\mu}{2}\right)\frac{\partial^2 v}{\partial x\partial y}=0 \quad (4)$$

$$\frac{\partial^2 v}{\partial y^2}+\left(\frac{1-\mu}{2}\right)\frac{\partial^2 v}{\partial x^2}+\left(\frac{1+\mu}{2}\right)\frac{\partial^2 u}{\partial x\partial y}=0 \quad (5)$$

Now the problem is to find u and v in a two dimensional field satisfying the two elliptic partial differential equations (4) and (5).

In the ψ formulation, the problem was reduced to the determination of a single function instead of two functions u and v, simultaneously, satisfying the equilibrium equations (4) and (5). In this formulation, the potential function ψ (x, y) was defined in terms of displacement components as

$$u=\frac{\partial^2\psi}{\partial x^2} \quad (6)$$

$$v=-\frac{1}{1+\mu}\left[(1-\mu)\frac{\partial^2\psi}{\partial y^2}+2\frac{\partial^2\psi}{\partial x^2}\right] \quad (7)$$

When the displacement components in equations (4) and (5) are replaced by ψ (x, y), equation (4) is automatically satisfied and the only condition that ψ has to satisfy becomes

$$\frac{\partial^4\psi}{\partial x^4}+2\frac{\partial^4\psi}{\partial x^2\partial y^2}+\frac{\partial^4\psi}{\partial y^4}=0 \quad (8)$$

The boundary conditions of the formulation were as follows

$$u(x,y)=\frac{\partial^2\psi}{\partial x^2} \quad (9)$$

$$\sigma_x(x,y)=\frac{E}{(1+\mu)^2}\left[\frac{\partial^3\psi}{\partial x^2\partial y}-\mu\frac{\partial^3\psi}{\partial y^3}\right] \quad (10)$$

$$v(x,y)=-\frac{1}{1+\mu}\left[(1-\mu)\frac{\partial^2\psi}{\partial y^2}+2\frac{\partial^2\psi}{\partial x^2}\right] \quad (11)$$

$$\sigma_y(x,y)=-\frac{E}{(1+\mu)^2}\left[\frac{\partial^3\psi}{\partial y^3}+(2+\mu)\frac{\partial^3\psi}{\partial x^2\partial y}\right] \quad (12)$$

$$\sigma_{xy}(x,y)=\frac{E}{(1+\mu)^2}\left[\mu\frac{\partial^3\psi}{\partial x\partial y^2}-\frac{\partial^3\psi}{\partial x^3}\right] \quad (13)$$

In Eq. (8) it is clear that the displacement components u and v and the stress components $\sigma_{x,}$, σ_y, σ_{xy} are not symmetric with respect to x and y axis. As a result the finite difference form of formula of these conditions give non-symmetric structure, which produce complexity in computer programming when a problem is solved in numerical approach. To eliminate this shortcoming a new formulation is proposed. The new formulation is described as follows.

2.2 New Formulation

In the proposed formulation, the problem is also reduced to the determination of a single function instead of two. In this formulation, a potential function ξ (x, y) is defined in terms of displacement components as

$$u=(1-\mu)\frac{\partial^2\xi}{\partial x^2}+2\frac{\partial^2\xi}{\partial y^2}-(1+\mu)\frac{\partial^2\xi}{\partial x\partial y} \quad (14)$$

$$v=(1-\mu)\frac{\partial^2\xi}{\partial y^2}+2\frac{\partial^2\xi}{\partial x^2}-(1+\mu)\frac{\partial^2\xi}{\partial x\partial y} \quad (15)$$

Here both u and v are symmetric with respect to x and y. When the displacement components in equations (4) and (5) are substitute by using equation (14) and (15) both the equations (4) and (5) transform to a single equation in the form as

$$\frac{\partial^4\xi}{\partial x^4}+2\frac{\partial^4\xi}{\partial x^2\partial y^2}+\frac{\partial^4\xi}{\partial y^4}=0 \quad (16)$$

The equation (16) is in the same form of Uddins formulation as shown in Eq^n (8). Therefore the problem is now formulated in such a way that a single function ξ has to be evaluated from the bi-harmonic equation (16), satisfying the boundary conditions that are specified at the boundary.

3. BOUNDARY CONDITION WITH ξ FORMULATION

The boundary conditions at any point on the rectangular boundary are known in terms of the rectangular components of displacement, u and v, and of stress, σ_x, σ_y, σ_{xy},

In order to solve the mixed boundary–value problems using the present formulation, the boundary conditions in terms of ξ becomes as follows

$$u(x,y)=(1-\mu)\frac{\partial^2\xi}{\partial x^2}+2\frac{\partial^2\xi}{\partial y^2}-(1+\mu)\frac{\partial^2\xi}{\partial x\partial y} \quad (17)$$

$$v(x,y)=(1-\mu)\frac{\partial^2\xi}{\partial y^2}+2\frac{\partial^2\xi}{\partial x^2}-(1+\mu)\frac{\partial^2\xi}{\partial x\partial y} \quad (18)$$

$$\sigma_x(x,y)=\frac{E}{1+\mu}\left[\frac{\partial^3\xi}{\partial x^3}-\frac{\partial^3\xi}{\partial x^2\partial y}+(2+\mu)\frac{\partial^3\xi}{\partial x\partial y^2}+\mu\frac{\partial^3\xi}{\partial y^3}\right] \quad (19)$$

$$\sigma_y(x,y)=\frac{E}{1+\mu}\left[\frac{\partial^3\xi}{\partial y^3}-\frac{\partial^3\xi}{\partial x\partial y^2}+(2+\mu)\frac{\partial^3\xi}{\partial x^2\partial y}+\mu\frac{\partial^3\xi}{\partial x^3}\right] \quad (20)$$

$$\sigma_{xy}(x,y)=\frac{E}{1+\mu}\left[\frac{\partial^3\xi}{\partial x^3}-\mu\frac{\partial^3\xi}{\partial x^2\partial y}-\mu\frac{\partial^3\xi}{\partial x\partial y^2}+\frac{\partial^3\xi}{\partial y^3}\right] \quad (21)$$

Here from Eqs. (17-21), it is observed that all the components of displacements and stresses are symmetric with respect to x and y axis.

4. FINITE DIFFERENCE FORM OF THE BOUNDARY VALUE FORMULAE

The discritized form of the boundary value formula for stresses $\sigma_{x,}$ σ_y and σ_{xy} from Eqs. (19-21) and for displacement u and v from Eq.(17-18) are shown in Fig. 1 and Fig. 2 respectively.

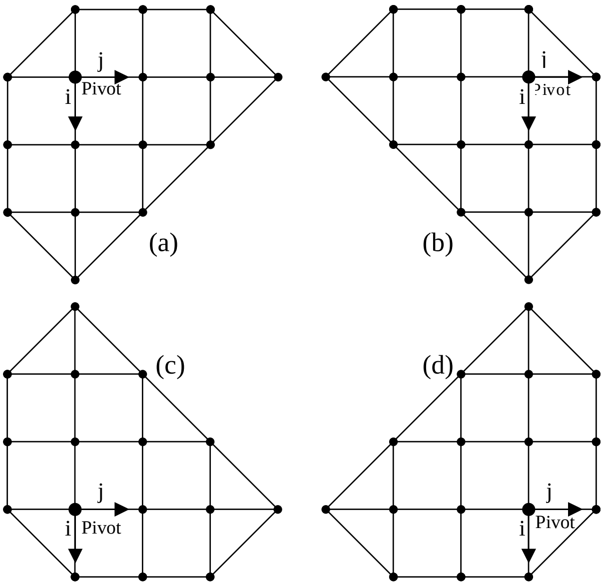

Fig 1. Grid structure of the boundary value formula for stresses for (a) Top-left boundary, both x and y forward difference, (b) Top-right boundary, x forward and y backward difference, (c) Bottom-left boundary, x backward and y forward difference and (d) Bottom-right boundary, both x and y backward difference.

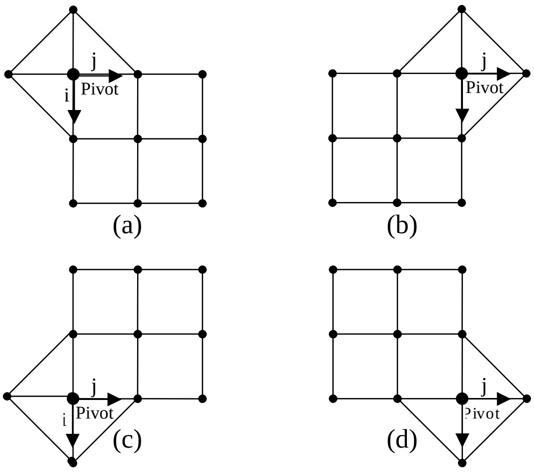

Fig 2. Grid structure of the boundary value formula for displacements (a) Top-left boundary, (b) Top-right boundary, (c) Bottom-left boundary, (d) Bottom-right boundary.

In our formulation the physical boundary has been divided into four segments. These segments are top- left, top-right, bottom-left and bottom right boundary. For each segment we have developed a particular formula structure that can be used on that particular boundary. Interesting thing is that all these structure are symmetric in nature. Thus we can use the formula structure repeatedly on all of the four boundaries. This eliminates the need of using different formula structure for the different boundary. This is our success over Uddin's formulation of using only one formula structure all over the boundary. Our new formulation lessens the complexity of computer programming. The advantage of using the formula structure over the boundary makes the computer code simple, easiest and of course quicker.

5. ELASTIC PROBLEM AND ITS BOUNDARY CONDITIONS

The geometry of the selected problem is shown in Fig. 3 (a). The dimensions a and b denote the length and height respectively. This is the problem of a deep beam under loading acting as a cantilever. Its one end is considered rigidly fixed, therefore, the boundary conditions are given by u=0, v=0 at the fixed end. Load is applied at the free end of the beam as shown in the figure.The other surfaces, those are free from loading, the boundary conditions are given as $\sigma_x/E=0$, $\sigma_y/E=0$, $\sigma_{xy}/E=0$.

To justify the validity of the proposed new formulation the selected problem is solved by developing a new computer code. Three specific sections are selected for analyzing the distribution of stress and displacement. The sections are shown in Fig. 3 (b). Stress analysis of these sections is described in article 7.

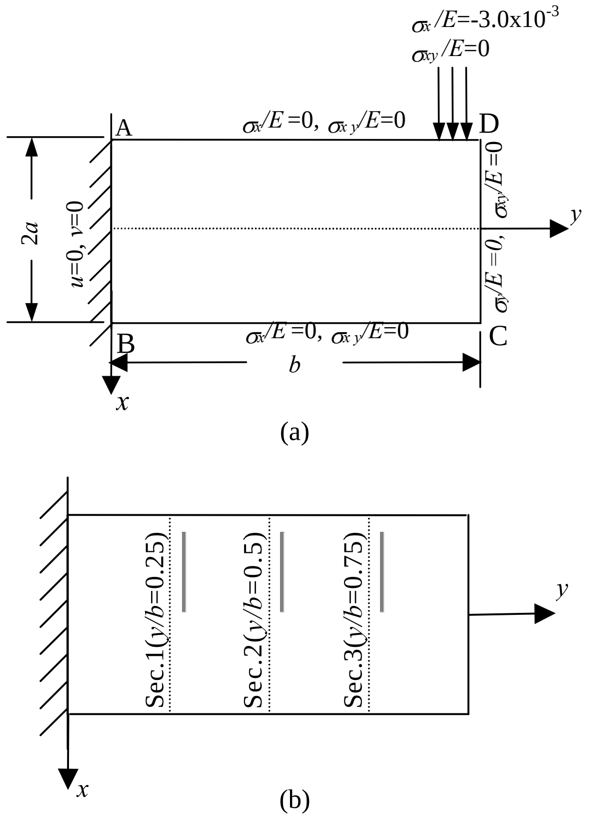

Fig 3. Geometry of the problem (a) boundary condition and (b) selected sections for stress analysis.

6. NUMERICAL SOLUTION

In practice the whole body of the bar is divided into a suitable number of rectangular mesh points as shown in Fig. 4. The mesh lengths are h (=0.25) and k (=0.5) in the x and y directions respectively. Considering mixed boundary value problem, the boundary conditions may be specified into four combinations as (u, v); (σ_y, σ_{xy}); (σ_x, σ_{xy}); (σ_{xy}, σ_x). These differential equations have been expressed in terms of finite difference form. The computer program is developed in such a way that out of two equations one is applied on the physical boundary grid point and the other is applied to the point exterior of the physical boundary. The discretized form of bi-harmonic equation is applied to interior points of the body. Thus the scheme provides a system of linear algebraic equations, which has been solved by lower-upper decomposition method.

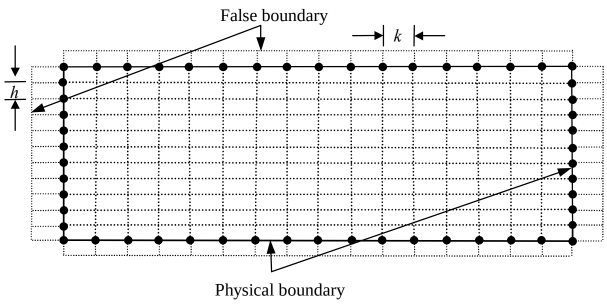

Fig 4. Rectangular meshing of the selected geometry.

7. RESULTS AND DISCUSSION

The displacement component, u as shown in Fig. 5 is in the direction of x axis. Theoretically the deflection should be greater as we travel from the rigid portion to the free end and also magnitude of deflection at any point on a section as presented in Fig. 3(b) should be same.

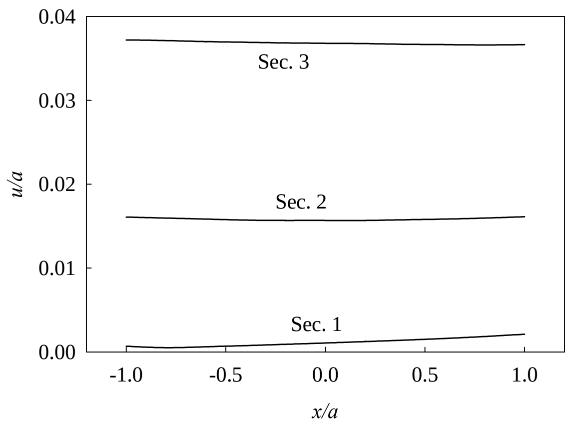

Fig 5. Distributaries of displacement component, u.

From our numerically solved distribution we see that the deflection is increasing from Sec.1 to Sec.3 carrying all the positive values. This enlightens our new formulation to be perfect.

The displacement component, v as shown in Fig. 6 is the component in the direction of y axis. Theoretically it is evident that under load the bar will buckle and the component in the direction of y axis will vary from positive value to zero and again from zero to negative value. Again buckling should be prominent as we travel from the rigid portion to the free end.

From our solution mentioned above, it is quite clear that this is the reflection of theoretical behavior. Experimentally as we go from Sec.1 to Sec.3 we see that the magnitude of displacement component is increasing. This indicates that the deflection goes from positive to zero value and then zero to a negative value and the deflection at the tip is the greatest. This characteristics proving that the new formulation is accurate.

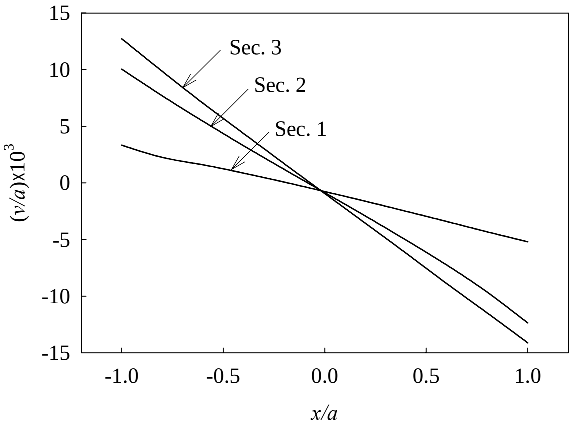

Fig 6. Distribution of displacement component, v.

The stress component, σ_y, as shown in Fig. 7 is the component of normal stress in the direction of y. Theoretically under the application of load the upper side of the bar should be in tension and the lower potion should be in compression due to bending moment. Again from the point of application of load as we go to the rigid portion the moment should be increased due to increasing in length. Thereby stress also increased from free end to the fixed end.

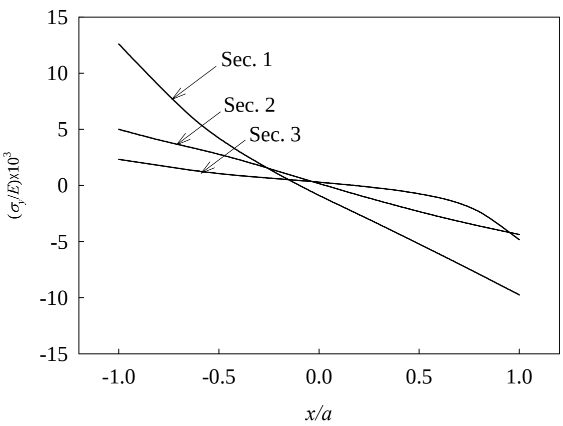

Fig 7. Distribution of stress component, σ_y.

Experimentally we see that the upper portion in tension and the lower portion in compression. And the stress in Sec.1 is greater than Sec.2 and Sec.3. Thus from the above point of view it can be clearly conclude that our new formulation is successful from each and every point of theoretical overview.

8. CONCLUSIONS

In this paper a new formulation is presented for the solution of mixed boundary elastic problem. The formula structures for stresses and displacements that are used here is symmetric in both x and y axes. Thus a single structure for all formula can be used in any of the four segments of the boundary by flipping this structure singly or doubly. For checking validity of new formulation, deep beam is solved. From the numerical solution we see that it satisfies the theoretical expectations. Finally, it can be conclude that this new formulation is valid for any complex boundary value problem.

9. REFERENCES

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10. NOMENCLATURE

Symbol	Meaning	Unit
h	Mesh length in x direction	(m)
k	Mesh length in y direction	(m)
E	Modulus of elasticity	(Pa)
$\psi(x,y)$	Displacement potential function	-
$\xi(x, y)$	Displacement potential function for the new formulation	-
μ	Poisson's ratio	-
u	Displacement component in x-direction	(m)
v	Displacement component in y-direction	(m)
σ_x	Stress component in x-direction	(Pa)
σ_y	Stress component in y-direction	(Pa)
σ_{xy}	Shear Stress component in xy-plane	(Pa)
a	Half height of deep beam	(m)
b	Length of deep beam	(m)