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EFFECT OF CIRCULAR HOLE ON THE DISTRIBUTION OF STRESSES IN A RECTANGULAR PLATE

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ABSTRACT

This paper presents the application of finite difference technique for the solution of an elastic problem of a rectangular body having internal circular hole. This study also focuses on the stress concentration developed at the critical zones around the circular hole of the body. A computer program has been developed to solve two dimensional plane strain problems using the formulations of displacement potential function. A new methodology is proposed to manage the internal boundary conditions of the hole. A model problem has been solved and the results are analyzed and compared with the solution available in the literature to justify the validity of the solution scheme.

Keywords: Finite difference technique, Stress analysis, Circular hole.

1. INTRODUCTION

Designing of machine components requires precise information of stress formulations over its entire body especially at the critical regions. Most of the practical problems appear with arbitrary shaped boundary and loading conditions may differ in problem to problem. But, stress analysis of elastic bodies of arbitrary shape is not easy task. The management of boundary shape and mixed boundary conditions is a difficult task for such bodies [1-4]. Elastic problems are formulated either in terms of deformation parameters or stress parameters. But, at the boundary, all the problems are invariably subjected to the mixture of both known deformations and stress. Neither of the two formulations would however allow us to account fully of both of these two types of boundary conditions with equal sophistication in the region of transition of boundary conditions from one type to another.

Displacement potential function formulation [5] is well known for the analysis of two-dimensional mixed boundary-value elastic problems. Using this formulation in conjunction with finite-difference technique Idris *et al.* [6-7] established the validity of the potential function by solving some simple problems. Later, a numerical model for solving simple rectangular body has been proposed by Ahmed *et al.* [8-11]. The accuracy and the reliability of the numerical model has been verified through subsequent advancements of the technique from rectangular to arbitrary shaped bodies [12-14]. In order to deal with the arbitrary shape of the boundary, the boundary values at point not coinciding with the rectangular nodal points are approximated by the linear interpolation of two or four neighboring nodal points.

The Numerical models in Finite Difference Technique developed so far are limited to the solution of solid elastic bodies i.e. body without any internal holes or flaws. Elastic body having internal hole or flaw poses a serious problem as it suffers stress concentration at the hole boundary. In finite difference technique, stress analysis of such a body becomes more complex due to management of boundary conditions at the external as well as at the internal boundary. In this paper, a numerical model has been proposed to solve problems of this nature. To examine the accuracy of the proposed method of solution in compared to the available theoretical and experimental results a rectangular plate having internal circular hole has been solved under tensile loading. The results are found in good agreement with the analytical solution.

2. DESCRIPTION OF THE PROBLEM AND BOUNDARY CONDITIONS

The geometry of the problem is shown in Fig 1. The geometry can be expressed as $b/a = 2.5$, $r/a = 0.25$.

2.1 Material

The material is assumed perfectly elastic and was given the property of material with Poisson's ratio ν is assumed 0.3. Despite the choice, this procedure is also valid for other materials.

2.2 Boundary Conditions

The boundary conditions of the chosen problem are shown in Fig 2. Boundary AB is considered rigidly fixed.

So there will be no displacement in this part of the boundary and thus the boundary conditions are set as u_n=0.0, u_t=0.0. At the right boundary (CD) uniform tensile load is applied. For this boundary for every nodal points the boundary conditions are set as $\sigma_n/E=3\times10^{-4}$, σ_t/E = 0.0, where the symbol E denotes modulus of elasticity.

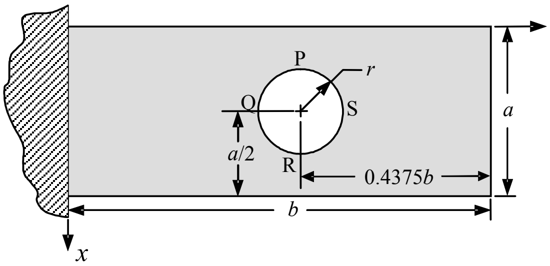

Fig 1. Geometry of the problem

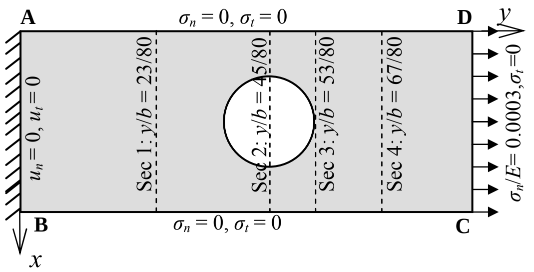

Fig 2. Boundary condition applied to the problem

The top and bottom boundaries (AD and BC) are free from stress. The boundary conditions for the free boundary are set as σ_n/E = 0.0, σ_t/E = 0.0

The surface of the internal hole is free from any external load; boundary conditions are, therefore, assigned as σ_n/E = 0.0, σ_t/E = 0.0.

3. EQUATIONS USED

3.1 Governing Equations

Stress analysis of an elastic body is usually a three dimensional problem. But, most of the practical problems appear usually in the state of plane stress or plane strain. Stress analysis of three-dimensional bodies under plane stress or plane strain can be treated as two-dimensional problems. The solution of two-dimensional problems requires the integration of the differential equations of equilibrium together with the compatibility equations and the boundary conditions. If body force is neglected, the equations to be satisfied are [1]

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = 0 \tag{1}$$

$$\frac{\partial \sigma_y}{\partial y} + \frac{\partial \sigma_{xy}}{\partial x} = 0 \tag{2}$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(\sigma_x + \sigma_y) = 0 \tag{3}$$

Substitution of stress components by displacement components u and v into Eqs. (1) to (3) makes Eq. (3) redundant and Eqs. (1) and (2) transform to

$$\frac{\partial^2 u}{\partial x^2} + \left(\frac{1-\nu}{2}\right)\frac{\partial^2 u}{\partial y^2} + \left(\frac{1+\nu}{2}\right)\frac{\partial^2 v}{\partial x \partial y} = 0 \tag{4}$$

$$\frac{\partial^2 v}{\partial y^2} + \left(\frac{1-\nu}{2}\right)\frac{\partial^2 v}{\partial x^2} + \left(\frac{1+\nu}{2}\right)\frac{\partial^2 u}{\partial x \partial y} = 0 \tag{5}$$

Now the problem is to find u and v from a two dimensional field satisfying the two elliptical partial differential Eqs. (4) and (5).

Instead of determining the two functions u and v the problem can be reduced to solving a single function $\psi(x, y)$, which can be determined by satisfying Eqs. (4) and (5). The displacement potential function $\psi(x, y)$ can be defined as [6]

$$u = \frac{\partial^2 \psi}{\partial x \partial y}$$
$$v = -\frac{1}{1+\nu}\left[(1-\nu)\frac{\partial^2 \psi}{\partial y^2} + 2\frac{\partial^2 \psi}{\partial x^2}\right] \tag{6}$$

By the above definitions the displacement components u and v satisfy Eq. (4) and the only condition reduced from Eq. (5) that the function $\psi(x, y)$ has to satisfy is

$$\frac{\partial^4 \psi}{\partial x^4} + 2\frac{\partial^4 \psi}{\partial x^2 \partial y^2} + \frac{\partial^4 \psi}{\partial y^4} = 0 \tag{7}$$

So, now the problem is to evaluate a single function $\psi(x, y)$ from the bi-harmonic Eq. (7), satisfying the boundary conditions specified at the boundary.

3.2 General Boundary Conditions

The boundary conditions at any point on an arbitrary-shaped boundary are known in terms of the normal and tangential components of displacement, u_n and u_t, and of stress, σ_n and σ_t. These four components are expressed in terms of u, v, σ_x, σ_y, σ_{xy} the components of displacement and stress with respect to the reference axes x and y of the body as follows:

$$u_n = u.l + v.m \tag{8}$$

$$u_t = v.l - u.m \tag{9}$$

$$\sigma_n = \sigma_x.l^2 + 2.\sigma_{xy}.l.m + \sigma_y.m^2 \tag{10}$$

$$\sigma_t = (l^2 - m^2)\sigma_{xy} + l.m(\sigma_y - \sigma_x) \tag{11}$$

At any point on the physical boundary, the boundary conditions are specified in terms of any two known values of u_n, u_t, σ_n and σ_t.

In order to solve the mixed boundary-value problems, all the boundary conditions are to be expressed in terms of function $\psi(x, y)$. The displacement components u_n, u_t as a function of $\psi(x, y)$ can be obtained by using Eqs. (6), (8) and (9). The stress components in Eqs. (10) and (11) can be obtained in terms of $\psi(x, y)$ using following expressions of rectangular stress components.

$$\sigma_x = \frac{E}{(1+\nu)^2}\left[\frac{\partial^3\psi}{\partial x^2 \partial y} - \nu\frac{\partial^3\psi}{\partial y^3}\right]$$

$$\sigma_y = -\frac{E}{(1+\nu)^2}\left[\frac{\partial^3\psi}{\partial y^3} + (2+\nu)\frac{\partial^3\psi}{\partial x^2 \partial y}\right] \qquad (12)$$

$$\sigma_{xy} = \frac{E}{(1+\nu)^2}\left[\nu\frac{\partial^3\psi}{\partial x \partial y^2} - \frac{\partial^3\psi}{\partial x^3}\right]$$

As far as numerical method of solution of equation (7) is concerned, it is evident from the expressions of boundary conditions (8) to (11) that, regardless of the combinations of two conditions specified on the boundary, the equations that $\psi(x, y)$ has to satisfy are Eq. (7) within the body and any two of the Eqs. (8) to (11) for points on the boundary. These equations are expressed as finite difference equations in terms of $\psi(x, y)$.

4. SOLUTION TECHNIQUE

For solution of the problem the elastic body under consideration is divided into a desirable number of rectangular meshes and the value of the function is sought only at these nodal or grid points of the mesh. The present program is to solve a function within a geometrically irregular region having internal hole. The region is divided into meshes with lines parallel to rectangular co-ordinate axes. As a result, the boundary may not pass through the rectangular nodal points. But the physical problems are associated with the known boundary conditions at the boundary points of irregular shaped elastic bodies, which require a further treatment to relate the values on the boundary with the nodal points. To overcome this problem a special technique is used. For designating any boundary point, a reference node point is used.

For any interior mesh point, it is seen that the bi-harmonic equation in terms of $\psi(x, y)$ applied to this point will give rise to a single algebraic equation and therefore, the single unknown concerning this point has been provided with a single equation for its evaluation. Further, this algebraic equation will contain the discretized variable of the thirteen neighbouring mesh points [8]. This implies that, for any mesh point, closest to the boundary mesh points, this equation will contain mesh points both interior and exterior to the boundary. Thus, to match the discretized bi-harmonic equation with the domain of field grid, at least one exterior point immediate vicinity of the boundary should be considered. Thus, it is seen that, if the domain is descretized by lines parallel to the rectangular co-ordinate system then the application of the finite difference formulae of the bi-harmonic equation places limitation to the points, immediate exterior neighbour-hood of the boundary mesh points. Again, the boundary conditions in terms of stress and displacement components contain 2nd and 3rd order derivatives of $\psi(x, y)$ and the application of the boundary conditions at an arbitrary point on the boundary will not be very easy without the involvement of exterior mesh points to the physical boundary of the domain concerned. Considering an arbitrary point on the boundary, the boundary conditions may be specified by any one of the four groups of boundary conditions, (u_n, u_t), (u_n, σ_t), (u_t, σ_n) and (σ_n, σ_t). Therefore, since the functions are not directly specified, there are always two conditions to be satisfied at an arbitrary point on the boundary and these two conditions are theoretically sufficient to provide two equations at this point. In this respect, if the boundary conditions are given either in terms of displacement or stress components, that is, in the form of differential equations of unknown function, these differential equations have to be expressed into difference equations.

As the differential equations associated with the boundary conditions contain second and third order derivatives of the function $\psi(x, y)$, the application of two-point forward or backward difference formula is required depending on the position of the boundary.

4.1 Transfer of Boundary Values to Grid Points and Placement of Boundary Conditions

The details description of the management of outer boundary points and their condition are given in our earlier studies [12-14]. Here the description of the management of internal hole boundary points are given as follows.

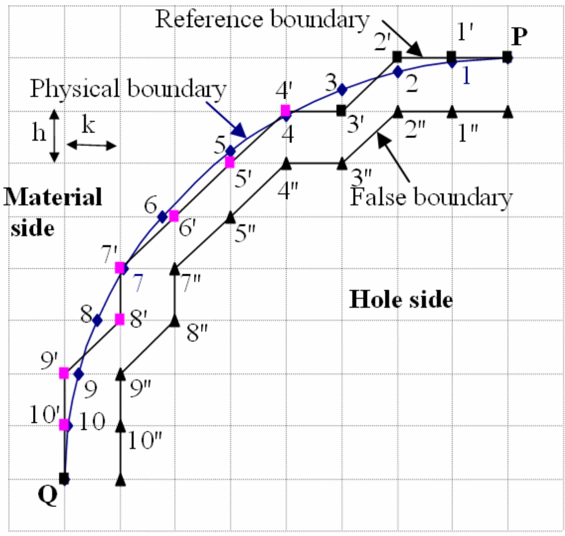

Fig 3. Management of hole boundary points

For describing the management of boundary points on the hole boundary, only the top-left portion, PQ (Fig 1) is selected and its details are presented in Fig. 3. If a physical boundary point match with a rectangular grid point then this point is considered as the reference point. But if physical boundary points do not match with the rectangular grid points then the grid points nearest to the physical boundary points are selected as the reference grid points. Here the points marked by 1, 2, 3 and 4 are the physical boundary points, and the corresponding reference boundary points are 1', 2', 3' and 4' respectively. So, 1'-2'-3'-4' is the reference boundary line. The false grid points are selected as 1", 2" and 3" thereby the false boundary line is 1"- 2"- 3"- 4". The physical boundary points on which this conditions are assigned are selected in such a fashion that for each boundary points there should be one point on reference boundary and one point on the false boundary.

As there are always two conditions to be satisfied at an arbitrary point on the physical boundary of the domain, two finite difference expressions of the differential equations associated with the boundary conditions are applied to the corresponding points on reference and false boundaries. A major problem is faced in formulating points on the sharp turning boundary, where a reference boundary point exists for which there is no corresponding point on false boundary. For such cases one or two conditions for a boundary point are found redundant. Such as in Fig. 3 for internal hole for reference boundary points 4', 5', 6' and 7', the corresponding reference points are 4", 5", 7" and 8" respectively. But, for reference point 6' there is no corresponding point on the false boundary. So, from the two conditions one is considered redundant. The treatment of boundary conditions as described using Fig. 3 covers particularly the portion of the top left boundary of circular hole. For the other part of the hole, the management of boundary conditions is done in similar fashion. A system of linear algebraic equations in terms of the function ψ has been obtained by the application of the finite difference formula on every mesh points within the elastic field. The LU-decomposition method has been used for solving the function ψ from the coefficient matrix developed by the system of linear equations. Thus the parameters of interest u, v, σ_x, σ_y and σ_{xy} at each nodal point in the body and u_n, u_t, σ_n, and σ_t at each point on the boundary are obtained from the solution of ψ.

5. RESULTS AND DISCUSSION

The solution of the displacement and stress components u, v, σ_x, σ_y and σ_{xy} for each rectangular nodal point within the elastic field has been obtained. Their distributions along some selected sections as shown in Fig. 2 are described below.

5.1 Distribution of *u*

The distribution of displacement component (along x axis) u for some selected sections as shown in Fig 4. It is observed that the values of displacement component are anti-symmetric with respect to the horizontal centerline of the body and changes sign from positive to negative at horizontal centerline (x/a=0.5). This clearly indicates the shortening of the dimension along x direction. Along the centerline there is no displacement along x-axis. It also complies with the known solution. The distribution of u for section 1 and section 4 coincides with each other proves the symmetrical distribution of u around the hole. The value of u is larger at the boundary and zero at horizontal centerline.

Displacement along section-2 across the hole is observed maximum. From solution it is found that the values of displacement component (u) of left most boundary are zero (not plotted here). The values of u for this boundary are set zero as boundary conditions. So, the solution satisfies the condition assigned on the left boundary. It also proves the accuracy of the solution.

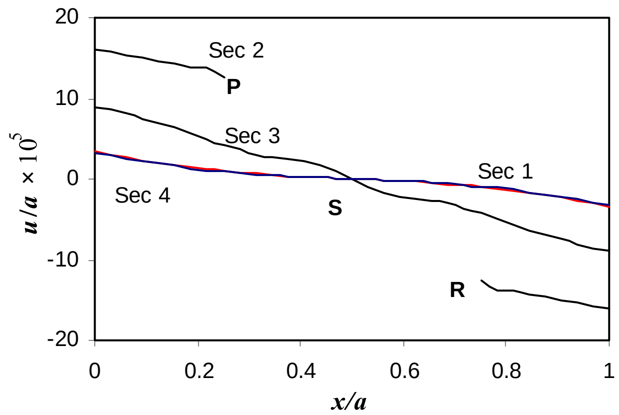

Fig 4. Distribution of u

5.2 Distribution of *v*

The Distributions of displacement component along y-axis v for different sections are shown in Fig 5. The plot shows the symmetrical distribution of v with respect to the horizontal centerline (x/a = 0.5). It is found that for any section between the hole and the loading boundary the value of v is large at the horizontal centerline.

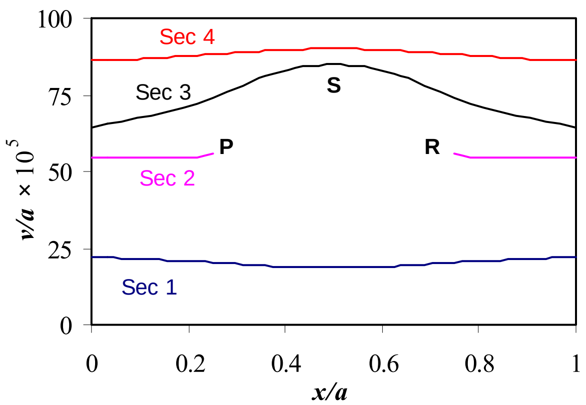

Fig 5. Distribution of v

For section in the other zone i.e. between the fixed boundary and hole, the value of v, is minimum at the horizontal centerline. Among the presented sections value of v is higher in section 4. It means that the displacement is higher in sections nearer to the force. In comparing the distribution of u (in Fig 4) and v (in Fig 5) it is found that v is very large in magnitude.

5.3 Distribution of σ_x

Distribution of stress components along x-axis is shown in Fig. 6. This figure shows that sections 1, 2 and 4 suffer tension along x-axis. Section 3, that passes the right boundary of the hole at point S, suffers compression. It also shows that at the hole-boundary at point S, the value of stress component is higher than any other points for any section. It proves the stress concentration phenomena along the hole-boundary.

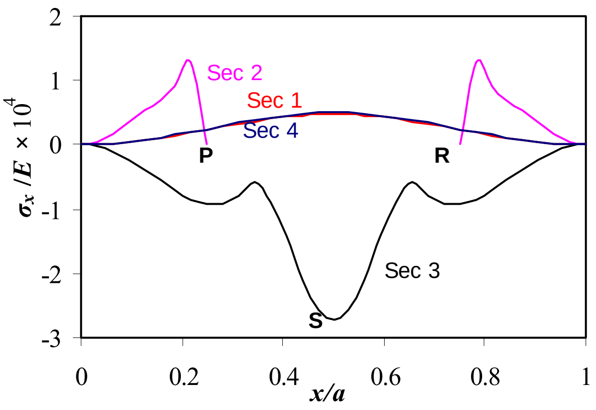

Fig 6. Distribution of σ_x

At the top and bottom boundaries (x/a = 0.0 and 1.0) the values of stress component are zero. It satisfies the boundary condition applied at those boundaries. Sections 1 and 4 show exactly the same distribution of stresses. Maximum compression is observed at point S (σ_x/E = -2.7×10^{-4}). At free boundary points (P and R) the values of σ_x are zero. If there is no hole inside the body then whole body will suffer compression along x axis. But due to presence of hole inside the body some portion of the body near the hole will suffer tension. This is due to the fact of flattening of the hole when load applied at the right end along y axis. The analytical result [1] shows that the value of σ_x at point S should be equal to the applied stress. Here in our solution from Fig 6 it can be found that the value of σ_x at point S is very near to 3×10^{-4}E (the applied stress).

5.4 Distribution of σ_y

Figure 7 shows the distribution of stress component (along y direction) σ_y. From the distribution of σ_y, it is observed that stress concentration occurs at points P and R of the hole-boundary.

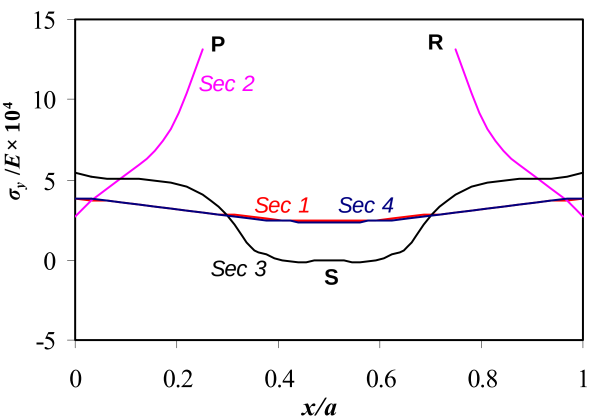

Fig 7. Distribution of σ_y

The distribution of stress component is symmetrical around the hole. It is found that in sections 1 and 4 the values of stress distribution are completely identical and are almost same as the applied stress. So, the distribution of stresses varies around the hole and becomes uniform large distance away from the hole. This fact satisfies the Saint-Venant's Principle.

Figure 7 shows that point P and R are the most critical point. According to the analytical results the stress developed at these points should be equal to 12.9×10^{-4}E, 4.3 times the applied stress. The result obtained from this scheme is, therefore, very nicely satisfied the analytical solution. Figure 7 shows that the stress developed at point S is zero which satisfies the free surface conditions.

6. CONCLUSIONS

The present technique of finite difference method provides analysis of stresses in a body having internal circular hole. The results found for hole boundary show good agreement with the analytical solution. This solution approach is also valid for any arbitrary shape of body having arbitrary shaped internal hole.

7. REFERENCES

1. S. Timoshenko, and J. N. Goodier, 1979, *Theory of Elasticity,* 3rd Ed., McGraw-Hill, New York, N. Y.
2. A. J. Durelli, and B. Ranganayakamma, 1989, "Parametric solution of stresses in beams", Journal of Engineering Mechanics, Vol. 115, No. 2, pp. 401.
3. J. O. Dow, M. S. Jones, and S. A. Harwood, 1990, "A new approach to boundary modeling for finite difference applications in solid mechanics", Int. J. for Num. Methods in Engineering, Vol. 30, pp. 99-113.
4. C. O. Horgan, and J. K. Knowels, 1983, "Recent development concerning Saint-Venant's principle", Adv. Appl. Mech., Vol. 23, pp. 179-269.
5. M. W. Uddin, 1966, "Finite Difference Solution of Two-Dimensional Elastic Problems with Mixed Boundary Conditions", M. Sc. Thesis, Carleton University, Canada.
6. A. B. M. Idris, 1993, "A New Approach to Solution of Mixed Boundary Value Elastic Problems", M. Sc. Thesis, Bangladesh University of Engineering and Technology, Dhaka, Bangladesh.
7. A. B. M. Idris, S. R. Ahmed and M. W. Uddin, 1996, "Analytical solution of a 2-D elastic problem with mixed boundary conditions", Journal of the Institution of Engineers (Singapore), Vol. 36 No. 6, pp. 11-17.
8. S.R. Ahmed, 1993, "Numerical Solutions of Mixed Boundary-Value Elastic Problems", M. Sc. Thesis, Bangladesh University of Engineering and Technology, Dhaka, Bangladesh.
9. S. R. Ahmed, M. R. Khan, K. M. S. Islam and M. W. Uddin, 1996, "Analysis of stresses in deep beams using displacement potential function", Journal of the Institution of Engineers (India), Vol. 77, pp. 141-147.
10. S. R. Ahmed, A. B. M. Idris and M. W. Uddin, "Numerical solution of both ends fixed deep beams", Computers and Structures, Vol. 61, 1996, pp. 21-29.
11. S. R. Ahmed, M. R. Khan, K. M. S. Islam and M. W. Uddin, 1998, "Investigation of stresses at the fixed end of deep cantilever beams", Computers and Structures, Vol. 69, pp. 329-338.

12. M. A. S. Akanda, 1997, “Stress Analysis of Gear Teeth”, M. Sc. Thesis, Bangladesh University of Engineering and Technology, Dhaka, Bangladesh.
13. M. A. S. Akanda, S. R. Ahmed, M. R. Khan and M.W. Uddin, 2000, “A finite difference scheme for mixed boundary value problem of arbitrary shaped elastic bodies”, Advances in engineering Software, Vol. 31, No. 3, pp. 173-184.
14. M. A. S. Akanda, S. R. Ahmed and M.W. Uddin, 2002, “Stress Analysis of gear teeth using displacement potential function and finite differences”, Int. Journal for Numerical Methods in Engineering, Vol. 53, pp. 1629-1640.
15. M. Arifur Rahman, and M. A. S. Akanda, 2004, “Finite Difference Scheme for Mixed Boundary Value Problem of Elastic Body with Internal Hole”, *Proceedings of 4th IMEC*, 29-31 December 2004, pp. 19-25

8. NOMENCLATURE

Symbol	Meaning	Unit
x, y	Rectangular co-ordinates	(mm)
E	Elastic modulus of the material	(Pa)
ν	Poisson's ratio	(mm/mm)
u, v	Displacement component in the x and y-direction	(mm)
σ_x, σ_y, σ_{xy}	Stress component in the x-direction, y-direction and xy-plane	(Pa)
ψ	Potential function defined in terms of displacements	-
u_n, u_t	Normal and tangential displacement component on the physical boundary	(mm)
σ_n, σ_t	Normal and tangential stress component on the physical boundary	(Pa)
l, m	Direction cosines of the normal at any point on the boundary	-
h, k	Mesh lengths in the x and y-directions	(mm)