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FINITE ELEMENT FREE VIBRATIONS OF LAMINATED COMPOSITE STIFFENED HYPAR SHELL ROOFS

Sarmila Sahoo and Dipankar Chakravorty

[1]Department of Civil Engineering, Jadavpur University, Kolkata – 700032, India

ABSTRACT

The skewed hypar shells are architecturally beautiful, easy to cast and are often preferred as roofing units. In the present study the literature on composite stiffened panel is studied and it is found that the free vibrations of composite stiffened hypar shell panels have not been studied in details. A finite element is developed combining a composite eight noded shell element with a three noded composite beam element. The code is validated by solving benchmark problems and fundamental frequencies are obtained for different combinations of laminations, boundary conditions and stiffener arrangements (both number and depth). Results are analysed thoroughly and the paper ends with a number of conclusions of design significance.

Keywords: Stiffened Hypar Shell, Free Vibration, Finite Element.

1. INTRODUCTION

Among the common civil engineering shell forms, which are used as roofing units, the skewed hypars have a special position because these architecturally pleasant forms may be cast and fabricated conveniently being doubly ruled surfaces. The hypar shells may be stiffened to have enhanced rigidity when subjected to point loads or provided with cutouts for some service requirements. A comprehensive idea about their static and free vibration characteristics is essential for a designer for successfully applying these forms. Nowadays researchers are emphasising more on laminated composite shells realising the strength and stiffness potentials of this advanced material.

The initial studies about vibrations of stiffened shell panels where about stiffened cylindrical shells reported from time to time by Bardell and Mead [1], Mecitoğlu and Dökmeci [2], Olson [3], Sinha and Mukhopadhyay [4], Jiang and Olson [5] who used different methods like collocation, finite strip and finite element. Sinha and Mukhopadhyay [6] echoed this fact in their review paper. As the researchers became more inclined towards composite materials a number of interesting papers came up dealing with free vibrations of stiffened composite shell panels most of which used the finite element as the analytical tool. Among these papers, Goswami and Mukhopadhyay [7, 8], Prusty and Satsangi [9] worked on both cylindrical shell and spherical shells while Rikards et al. [10] took up cylindrical stiffened shell panels. Recently Nayak and Bandyopadhyay [11, 12] carried out free vibration studies of isotropic stiffened shell panels in details including stiffened hypar shells. Free and forced vibrations of unstiffened composite hypar shell was reported by Chakravorty et al. [13]. In a recent paper Sahoo and Chakravorty [14] presented results of static analysis of composite hypar shells but without stiffeners.

An overall look at the volume of literature that has accumulated till date dealing with stiffened shell panels reflect the fact that stiffened composite skewed hypar shells have not received due attention by researchers. This, no doubt, defines a wide area of research and the present paper aims to focus on the free vibration characteristics of stiffened composite hypar shells.

2. MATHEMATICAL FORMULATION

2.1 Finite Element Formulation for Shell

A laminated composite hypar shell of uniform thickness h and twist radius of curvature R_{xy} is considered. Keeping the total thickness same, the thickness may consist of any number of thin laminae each of which may be arbitrarily oriented at an angle θ with reference to the x-axis of the co-ordinate system. An eight-noded curved quadratic isoparametric finite element (Fig.1) is used for hypar shell analysis. The five degrees of freedom taken into consideration at each node are u, v, w, α, β. Sahoo and Chakravorty [14] reported the strain displacement and constitutive relationships together with the systematic development of stiffness matrix for the shell element.

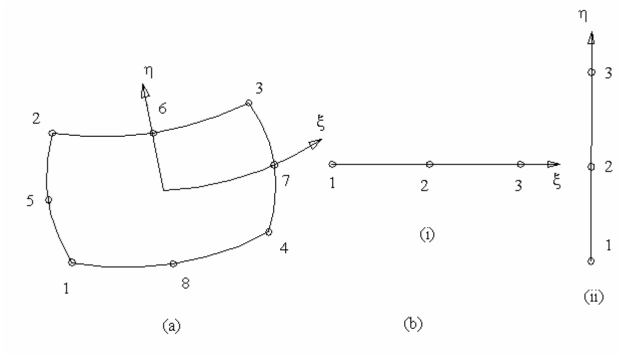


Fig 1(a). Eight-noded shell element with isoparametric co-ordinates (b) Three noded stiffener elements (i) *x*-stiffener (ii) *y*- stiffener

2.2 Finite Element Formulation for Stiffener Of The Shell

Three noded curved isoparametric beam elements (Fig.1) are used to model the stiffeners, which are taken to run only along the boundaries of the shell elements. In the stiffener element, each node has four degrees of freedom i.e. u_{sx}, w_{sx}, α_{sx} and β_{sx} for *x*-stiffener and u_{sy}, w_{sy}, α_{sy},and β_{sy} for *y*-stiffener. The generalized force-displacement relation of stiffeners can be expressed as:

x-stiffener: $\{F_{sx}\} = [D_{sx}]\{\varepsilon_{sx}\} = [D_{sx}][B_{sx}]\{\delta_{sxi}\}$;
y-stiffener: $\{F_{sy}\} = [D_{sy}]\{\varepsilon_{sy}\} = [D_{sy}][B_{sy}]\{\delta_{syi}\}$ (1)

where, $\{F_{sx}\} = [N_{sxx} \quad M_{sxx} \quad T_{sxx} \quad Q_{sxxz}]^T$; $\{\varepsilon_{sx}\} = [u_{sx.x} \quad \alpha_{sx.x} \quad \beta_{sx.x} \quad (\alpha_{sx} + w_{sx.x})]^T$

and $\{F_{sy}\} = [N_{syy} \quad M_{syy} \quad T_{syy} \quad Q_{syyz}]^T$; $\{\varepsilon_{sy}\} = [v_{sy.y} \quad \beta_{sy.y} \quad \alpha_{sy.y} \quad (\beta_{sy} + w_{sy.y})]^T$

Elasticity matrices are as follows:

$$[D_{sx}] = \begin{bmatrix} A_{11}b_{sx} & B'_{11}b_{sx} & B'_{12}b_{sx} & 0 \\ B'_{11}b_{sx} & D'_{11}b_{sx} & D'_{12}b_{sx} & 0 \\ B'_{12}b_{sx} & D'_{12}b_{sx} & \frac{1}{6}(Q_{44}+Q_{66})d_{sx}b_{sx}^3 & 0 \\ 0 & 0 & 0 & b_{sx}S_{11} \end{bmatrix}$$

$$[D_{sy}] = \begin{bmatrix} A_{22}b_{sy} & B'_{22}b_{sy} & B'_{12}b_{sy} & 0 \\ B'_{22}b_{sy} & \frac{1}{6}(Q_{44}+Q_{66})b_{sy} & D'_{12}b_{sy} & 0 \\ B'_{12}b_{sy} & D'_{12}b_{sy} & D'_{11}d_{sy}b_{sy}^3 & 0 \\ 0 & 0 & 0 & b_{sy}S_{22} \end{bmatrix}$$

where, $D'_{ij} = D_{ij} + 2eB_{ij} + e^2 A_{ij}$; $B'_{ij} = B_{ij} + eA_{ij}$, (2)

Here the shear correction factor is taken as 5/6. The sectional parameters are calculated with respect to the mid-surface of the shell by which the effect of eccentricity of the *x*-stiffener, e_{sx} and *y*-stiffener, e_{sy} are automatically included. The element stiffness matrix:

for *x*-stiffener: $[K_{xe}] = \int [B_{sx}]^T [D_{sx}][B_{sx}]dx$;

for *y*-stiffener: $[K_{ye}] = \int [B_{sy}]^T [D_{sy}][B_{sy}]dy$ (3)

The integrals are converted to isoparametric coordinates and are carried out by 2 point Gaussian quadrature. Finally, the element stiffness matrix of the stiffened shell is obtained by appropriate matching of the nodes of the stiffener and shell elements through the connectivity matrix and is given as:

$$[K_e] = [K_{she}] + [K_{xe}] + [K_{ye}]. \quad (4)$$

The element stiffness matrices are assembled to get the global matrices.

2.3 Element Mass Matrix

The element mass matrix for shell is obtained from the integral

$$[M_e] = \iint [N]^T [P][N]dxdy , \quad (5)$$

where,

$$[N] = \sum_{i=1}^{8} \begin{bmatrix} N_i & 0 & 0 & 0 & 0 \\ 0 & N_i & 0 & 0 & 0 \\ 0 & 0 & N_i & 0 & 0 \\ 0 & 0 & 0 & N_i & 0 \\ 0 & 0 & 0 & 0 & N_i \end{bmatrix}$$

$$[P] = \sum_{i=1}^{8} \begin{bmatrix} P & 0 & 0 & 0 & 0 \\ 0 & P & 0 & 0 & 0 \\ 0 & 0 & P & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \end{bmatrix},$$

in which

$$P = \sum_{k=1}^{n} \int_{z_{k-1}}^{z_k} \rho dz \text{ and } I = \sum_{k=1}^{n} \int_{z_{k-1}}^{z_k} z\rho dz \quad (6)$$

Element mass matrix for stiffener element

$$[M_{sx}] = \iint [N]^T [P][N]dx \quad \text{for x stiffener and}$$

$$[M_{sy}] = \iint [N]^T [P][N]dy \quad \text{for y stiffener} \quad (7)$$

Here, $[N]$ is a 3x3 diagonal matrix.

for *x*-stiffener

$$[P] = \sum_{i=1}^{3} \begin{bmatrix} \rho.w_{st}d_{sx} & 0 & 0 & 0 \\ 0 & \rho.w_{st}d_{sx} & 0 & 0 \\ 0 & 0 & \rho.w_{st}d_{sx}^2/12 & 0 \\ 0 & 0 & 0 & \rho(w_{st}.d_{sx}^3 + w_{sx}^3.d_{sx})/12 \end{bmatrix}$$

for *y* stiffener

$$[P] = \sum_{i=1}^{3} \begin{bmatrix} \rho.w_{sy}d_{sy} & 0 & 0 & 0 \\ 0 & \rho.w_{sy}d_{sy} & 0 & 0 \\ 0 & 0 & \rho.w_{sy}d_{sy}^2/12 & 0 \\ 0 & 0 & 0 & \rho(w_{sy}.d_{sy}^3 + w_{sy}^3.d_{sy})/12 \end{bmatrix}$$

Finally, the element mass matrix of the stiffened shell is obtained by appropriate matching of the nodes of the stiffener and shell elements through the connectivity matrix and is given as:

$$[M_e] = [M_{she}] + [M_{xe}] + [M_{ye}]. \quad (8)$$

The element mass matrices are assembled to get the global matrices.

2.4 Solution Procedure for Free Vibration Analysis

The free vibration analysis involves determination of natural frequencies from the condition

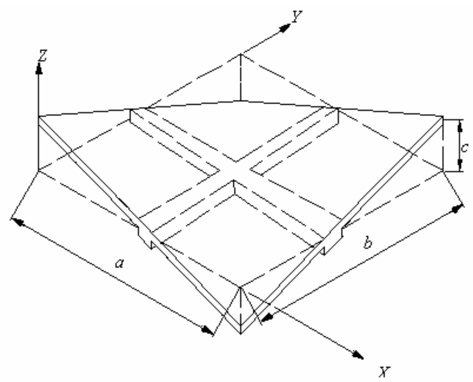
$$\left| [K] - \omega^2 [M] \right| = 0 \quad (9)$$

This is a generalized eigenvalue problem and is solved by the subspace iteration algorithm.

2.5 Numerical Examples

A simply supported square plate with one stiffener in one plan direction is analysed applying the present formulation making the rise of the hypar shell zero. The comparison of fundamental frequency obtained by Mukherjee and Mukhopadhyay [1988], Nayak and Bandyopadhyay [12] and present method is presented in Table 1. Further a comparison of the nondimensional fundamental frequencies of cantilever twisted plates obtained by Qatu and Leissa[(1991] and those by present method is presented in Table 2.

Additional problems of stiffened skewed hypar shells (Fig. 2) are solved for eight different types of stacking sequences of shell surfaces, two different types of boundary conditions, different types of stiffening schemes and different stiffener to shell thickness ratio with graphite-epoxy as the material. The individual lamina properties are assumed to be as E_{11}=25E_{22}, G_{12} =G_{13} =0.5E_{22}, G_{23} =0.2E_{22}, ν_{12} = ν_{21} =0.25. However, in all the cases the fibers in the stiffeners are considered to be arranged in a single layer along the length. The fundamental frequencies of different combinations for lamination, boundary conditions, stiffening schemes and different stiffener to shell thickness ratios are presented in Tables 3-4.



Surface equation: $z = \frac{4c}{ab}(x - a/2)(y - b/2)$

Fig 2. A typical skewed hypar shell panel with biaxial stiffeners eccentric at shell bottom

3. RESULTS AND DISCUSSION

The results of Table 1 show that the agreement of present results with the earlier ones is excellent and the correctness of the free vibration formulation of stiffened hypar shell is established. The fundamental frequencies of cantilever twisted plates obtained by Qatu and Leissa [1991] compare well with the present results as shown in Table 2 and the correct incorporation of twist of curvature in the formulation is established.

The converged values of the nondimensional fundamental frequencies of the authors' own problems are presented in Tables 3-4. Fundamental frequency is taken to have converged for particular finite element grid, if further refinement of the grid does not improve that result by more than one percent. With this criterion a 12x12 mesh is found to be appropriate for all the problems taken up here.

3.1 Free Vibration Response of Bare and Stiffened Hypars Combining Different Laminations and Boundary Conditions

Nondimensional fundamental frequencies for composite hypar shells with bare and stiffened surfaces are furnished in Tables 3 for simply supported and clamped boundary conditions respectively. The laminations include two, three and four layered antisymmetric and symmetric, cross and angle plies. For this preliminary study only central stiffeners are considered running along either one or both of the plan directions. For all the laminations frequencies of *X*-stiffened and *Y*-stiffened shells are comparable for clamped boundary conditions. The same trend is true for simply supported boundary also saving the cases of three and four layered symmetric cross ply laminates where *X*-stiffeners are found to impart considerably greater dynamic rigidity compared to the *Y*-stiffeners. In all the cases, however, a biaxially stiffened shell has greater frequency than a shell with a single stiffener. It is further noted that for cross ply clamped shells the increase of frequency of a bare shell on stiffening is insignificant. An overall study of Tables 3 reveals that the angle ply laminates are better than the cross ply ones for both the boundary conditions. Four layered symmetric and antisymmetric laminates appear to be the best choices of stacking orders for SSSS and CCCC stiffened surfaces respectively.

3.2 Relative Performances of Stiffened Hypars Combining Different Stiffener Depths, Laminations and Boundary Conditions

In practical situations where an engineer has the flexibility of using one or more stiffeners in either directions, he may be interested in knowing the stiffener arrangement that will yield the highest frequency for a given amount of material consumption. Table 4 furnishes fundamental frequencies of shell with the number of stiffeners in either direction as 1, 3 and 5 with the corresponding d_{st}/h ratio as 5, 1.6 and 1. Hence each of these options utilizes the same quantity of material. The close perusal of the result brings out the fact that using a single set of biaxial deep stiffeners (n_x=n_y=1 and d_{st}/h=5) is a better choice compared to the use of a number of shallow stiffeners in 13 out of 16 combinations of laminations and boundary conditions considered here. The exception to this general tendency are noted for antisymmetric angle ply laminates and for simply supported antisymmetric cross ply laminate. Hence one can confidently infer that for three or four layered angle and cross ply shells using a set of deep biaxial central stiffeners may be recommended. If one uses two layered laminates he has to try out the stiffest configuration by varying number and depth of stiffeners.

4. CONCLUSIONS

The present study leads to the following conclusions.

1. The composite stiffened shell element used here is suitable for analysing free vibration problems of composite stiffened hypar shells.
2. For clamped hypar shells the performance of X and Y stiffeners are comparable.
3. For simply supported boundary condition, however, some exceptions to this general trend are noted for three and four layered symmetric cross ply laminates
4. Bare cross ply clamped shells hardly undergo any improvement of frequency on stiffening.
5. For both the boundary conditions angle ply laminates are better choices than the cross ply one in terms of fundamental frequencies. For simply supported boundary condition the four layered symmetric angle ply is the best laminate while for clamped shells the four layered antisymmetric lamination exhibit the best performance.
6. If a designer has the flexibility of varying the number and depth of stiffener keeping the material consumption as constant, he should opt for a single set of biaxial deep stiffeners for three and four layered laminates. This recommendation holds good for both simply supported and clamped shells.

Table 1: Natural frequencies (*Hz*) of centrally stiffened clamped square plate

Mode no.	Mukherjee and Mukhopadhyay (1988)	Nayak and Bandyopadhyay(2002b)		Present method
		N8 (FEM)	N9 (FEM)	
1	711.8	725.2	725.1	733

a=b=0.2032 m, h=0.0013716 m, d_{st}= 0.0127 m, w_{st} =0.00635 m, stiffener eccentric at bottom, Material property: E=6.87x10^{10} N/m^2, ν=0.29, ρ=2823 kg/m^3

Table 2: Nondimensional natural frequencies ($\bar{\omega}$) for three layer graphite epoxy twisted plates, [θ/-θ/θ] laminate

Angle of twist	θ (degree)	0	15	30	45	60	75	90
ϕ=15^0	Qatu and Leissa(1991)	1.0035	0.9296	0.7465	0.5286	0.3545	0.2723	0.2555
	Present FEM	0.9989	0.9258	0.7443	0.5278	0.3541	0.2720	0.2551
ϕ=30^0	Qatu and Leissa(1991)	0.9566	0.8914	0.7205	0.5149	0.3443	0.2606	0.2436
	Present FEM	0.9491	0.8840	0.7181	0.5141	0.3447	0.2614	0.2445

a/b=1, a/h=100; E_{11}=138 GPa, E_{22}=8.96 GPa, G_{12}=7.1 GPa, ν_{12}=0.3

Table 3: Non dimensional fundamental frequency of simply supported (SSSS) and clamped (CCCC) shell

	SSSS				CCCC			
Laminations (Degree)	n_x=0, n_y=0	n_x=1, n_y=0	n_x=0, n_y=1	n_x=1, n_y=1	n_x=0, n_y=0	n_x=1, n_y=0	n_x=0, n_y=1	n_x=1, n_y=1
0/90	6.04644	6.10935	6.0659	7.60834	17.2268	17.5550	17.5466	17.8214
0/90/0	6.47008	8.73346	6.47307	8.74104	17.6786	17.9648	17.8316	18.2082
0/90/0/90	7.72463	7.77772	7.72968	9.27972	17.6069	17.9067	17.9067	18.1616
0/90/90/0	6.92198	8.96514	6.92501	9.02681	17.7269	18.0191	17.9912	18.2584
+45/-45	5.97132	5.99681	5.97797	7.62406	18.2576	19.5535	19.4445	25.5791
+45/-45/+45	8.53660	8.76874	8.80660	10.1723	21.6202	23.4061	23.3884	28.4175
+45/-45/+45/-45	8.54295	8.54607	8.55353	10.1763	21.6021	24.1832	24.1341	29.7274
+45/-45/-45/+45	8.9652	9.0858	9.1229	10.5489	21.8578	23.9762	23.9714	28.7803

a/b=1, a/h=100, c/a=0.2, w_{st}/h=1, d_{st}/h=2; E_{11} =25E_{22} ,G_{12} =G_{13} =0.5E_{22}, G_{23} =0.2E_{22}, ν_{12} = ν_{21} =0.25

Table 4: Non dimensional fundamental frequency of simply supported (SSSS) and clamped (CCCC) shell for different d_{st}/h ratio.

Laminations (Degree)	SSSS			CCCC		
	n_x=1, n_y=1	n_x=3, n_y=3	n_x=5, n_y=5	n_x=1, n_y=1	n_x=3, n_y=3	n_x=5, n_y=5
	d_{st}/h=5	d_{st}/h=1.6	d_{st}/h=1	d_{st}/h=5	d_{st}/h=1.6	d_{st}/h=1
0/90	8.9584	9.1596	9.0745	19.3797	17.8918	17.1713
0/90/0	10.7543	10.2971	9.5433	20.4062	18.2367	17.5332
0/90/0/90	11.0579	10.7069	10.2860	20.3920	18.1855	17.4740
0/90/90/0	10.9887	10.5424	9.8177	20.4779	18.2747	17.5711
+45/-45	9.1265	9.3041	9.3214	28.1148	29.2387	29.0199
+45/-45/+45	12.2741	11.6139	11.1985	30.9555	30.5244	30.7596
+45/-45/+45/-45	12.3645	11.6643	11.1815	32.5007	31.2921	31.5851
+45/-45/-45/+45	12.7093	11.0225	11.4663	32.2231	31.3933	31.7606

a/b=1, a/h=100, c/a=0.2, w_{st}/h=1; E_{11} =25E_{22} ,G_{12} =G_{13} =0.5E_{22}, G_{23} =0.2E_{22}, ν_{12} = ν_{21} =0.25

5. REFERENCES

1. Bardell, N.S. and Mead, D.J., 1989 "Free vibration of an orthogonally stiffened cylindrical shell, part II: discrete general stiffeners", Journal of Sound and Vibration, 134(1) : 55-72.
2. Mecitoğlu, Z. and Dökmeci, M.C., 1991, "Free vibrations of a thin, stiffened, cylindrical shallow shell", AIAA Journal, 30(3) : 848-850.
3. Olson, M.D., 1991 "Efficient modelling of blast loaded stiffened plate and cylindrical shell structures", Computers & Structures, 40(5) : 1139-1149.
4. Sinha, G. and Mukhopadhyay, M., 1994, "Finite element free vibration analysis of stiffened shells", Journal of Sound and Vibration, 171(4) : 529-548.
5. Jiang, J. and Olson, M.D., 1994 "Vibration analysis of orthogonally stiffened cylindrical shells using super finite elements", Journal of Sound and Vibration, 173(1) : 73-83.
6. Sinha, G. and Mukhopadhyay, M., 1995, "Static and dynamic analysis of stiffened shells-A review", Proc. Indian Natn. Sci. Acad., 61A(3 & 4) : 195-219.
7. Mukhopadhyay, M. and Goswami, S., 1996, "Transient finite element dynamic response of laminated composite stiffened shell", Journal of the Royal Aeronautical Society, June/July : 223-233.
8. Goswami, S. and Mukhopadyyay, M., 1995, "Finite element free vibration analysis of laminated composite stiffened shell", Journal of Composite Materials, 29(18) : 2388-2422.
9. Prusty, B.G. and Satsangi, S.K., 2001, "Finite element transient dynamic analysis of laminated stiffened shells", Journal of Sound and Vibration, 248(2) : 215-233.
10. Rikards, R., Chate, A. and Ozolinsh, O., 2001, "Analysis for buckling and vibrations of composite stiffened shells and plates", Composite Structures, 51 : 361-370.
11. Nayak, A.N. and Bandyopadhyay, J.N., 2002a, "Free vibration analysis and design aids of stiffened conoidal shells", Journal of Engineering Mechanics, 128(4) : 419-427.
12. Nayak, A.N. and Bandyopadhyay, J.N., 2002b, "On the free vibration of stiffened shallow shells", Journal of Sound and Vibration, 255(2) : 357-382.
13. Chakravorty, D., Bandyopadhyay, J.N. and Sinha, P.K., 1998, "Applications of FEM on free and forced vibrations of laminated shells", ASCE Journal of Engineering Mechanics, 124 (1) : 1-8.
14. Sahoo, S. and Chakravorty, D., 2004, "Finite element bending behaviour of composite hyperbolic paraboloidal shells with various edge conditions", Journal of Strain Analysis for Engineering Design, 39(5) : 499-513.

6. ACKNOWLEDGEMENT

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7. NOMENCLATURE

Symbol	Meaning
a,b, c	length, width and rise of shell
b_{sx}, b_{sy}	width of x and y stiffener
d_{sx}, d_{sy}	depth of x and y stiffener
d_{st}	depth of stiffener
D	elasticity matrix
E_{11},E_{22}	elastic moduli
G_{12},G_{13}, G_{23}	shear moduli of a lamina
np	number of plies in a laminate
n_x,n_y	no of stiffeners in x and y directions
w_{st}	width of stiffener
z_k	distance of bottom of kth ply from mid-surface of a laminate
ν_{12}, ν_{21}	Poisson's ratios
ρ	density of material
ω	natural frequency