Proceedings of the
International Conference on Mechanical Engineering 2005
(ICME2005) 28- 30 December 2005, Dhaka, Bangladesh

ICME05-AM-26

DISPLACEMENT POTENTIAL SOLUTION OF A DEEP STIFFENED CANTILEVER BEAM OF COMPOSITE MATERIALS

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ABSTRACT

In this paper, a deep stiffened cantilever of orthotropic composite materials is considered in order to analyze the elastic field due to a parabolic shear loading at the tip. Following the new approach of displacement potential, the mixed-boundary-value elastic problem is formulated in terms of a single displacement potential function. The solutions are obtained in the form of an infinite series. Some numerical results of different stress and displacement components at different critical sections of the deep cantilever beam are presented in the form of graphs. The results appear to be quite reasonable and accurate, and thus establish the soundness as well as reliability of the present displacement potential approach.

Keywords: Analytical solution, Stiffened cantilever beam, Displacement potential function.

1. INTRODUCTION

Elasticity problems are usually formulated either in terms of deformation parameters or stress parameters. Among the existing mathematical models of plane boundary-value stress problems, the stress function approach [1] and the displacement formulation [2] are noticeable. Successful application of the stress function formulation in conjunction with finite-difference technique has been reported for the solution of plane elastic problems where all the conditions on the boundary are prescribed in terms of stresses only [3, 4]. Further, Conway and Ithaca [5] extended the stress function formulation in the form of Fourier integrals to the case where the material is orthotropic, and obtained analytical solutions for a number of ideal problems. The shortcoming of the stress function approach is that it accepts boundary conditions only in terms of loadings. Boundary restraints specified in terms of the displacement components cannot be satisfactorily imposed on the stress function. As most of the practical problems of elasticity are of mixed boundary conditions, the stress function approach fails to provide any explicit understanding of the state of stresses at the critical regions of supports and loadings. The displacement formulation, on the other hand, involves finding two displacement functions simultaneously from the two second-order elliptical partial differential equations of equilibrium, which is extremely difficult, and this problem becomes more serious when the boundary conditions are mixed [2]. The difficulties involved in trying to solve practical stress problems using the existing models are clearly pointed out by Durelli and Ranganayakuma [6].

As stated above, neither of the formulations is suitable for solving problems of mixed boundary conditions, and hence a new mathematical model is used to solve the present problem of composite structure. In this approach, the plane elastic problem is formulated in terms of a single potential function of space variables. It should be mentioned that the present modeling approach enables us to manage the mixed mode of the boundary conditions as well as their zones of transition very efficiently. The present paper demonstrates the application of the displacement potential approach to the analytical solution of a deep stiffened cantilever beam of orthotropic composite material subjected to a parabolic shear loading. The supporting edge of the beam is assumed to be rigidly fixed and the two opposing edges are stiffened. The solutions are obtained in the form of an infinite series and the corresponding distributions of different stress and displacement components are presented mainly in the form of graphs.

2. ANALYTICAL MODEL OF THE PROBLEM

With reference to the Cartesian coordinate system x–y, a deep stiffened cantilever beam of composite materials is shown in Fig. 1. The fibers are directed along the length of the beam. The left edge is rigidly fixed to a support and the opposing edges are stiffened. The height and the length of the beam are designated by a and b, respectively. The tip of the beam is subjected to a parabolic shear load σ_{xy}, which is a function of y only.

For this model of the problem, different stress and displacement components are calculated at different critical sections of the beam using the method of single displacement potential function.

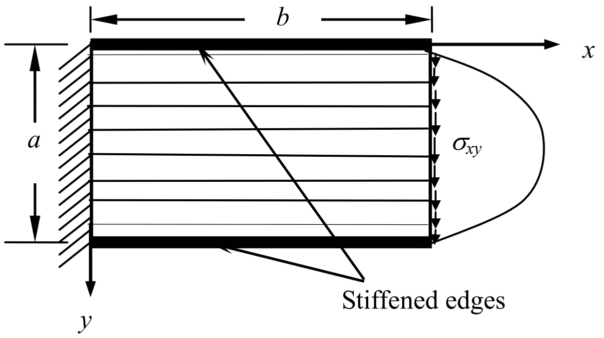

Fig 1. Analytical model of the problem

3. DISPLACEMENT POTENTIAL FORMULATION FOR THE PROBLEM

With reference to a rectangular Cartesian coordinate system and in the absence of body forces, the equilibrium equations are given by [1]

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = 0 \tag{1a}$$

$$\frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{xy}}{\partial x} = 0 \tag{1b}$$

To express the equilibrium equations in terms of displacement components, we need to express the three stress components in terms of displacement parameters. The corresponding three stress-displacement relations for general orthotropic materials are obtained from the Hooke's law as follows [7]

$$\sigma_{xx} = \frac{E_1}{1-\mu_{12}\,\mu_{21}}\left[\frac{\partial u_x}{\partial x} + \mu_{21}\frac{\partial u_y}{\partial y}\right] \tag{2a}$$

$$\sigma_{yy} = \frac{E_2}{1-\mu_{12}\,\mu_{21}}\left[\frac{\partial u_y}{\partial y} + \mu_{12}\frac{\partial u_x}{\partial x}\right] \tag{2b}$$

$$\sigma_{xy} = G_{12}\left[\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right] \tag{2c}$$

Substituting the above stress-displacement relations into Eqs. (1a) and (1b) and using the reciprocal relation $E_2\,\mu_{12} = E_1\,\mu_{21}$, we obtain the two equilibrium equations for two-dimensional problems of orthotropic materials in terms of the two displacement components as

$$\left(\frac{E_1^2}{E_1-\mu_{12}^2E_2}\right)\frac{\partial^2 u_x}{\partial x^2} + \left(\frac{\mu_{12}E_1E_2}{E_1-\mu_{12}^2E_2}+G_{12}\right)\frac{\partial^2 u_y}{\partial x\,\partial y} + G_{12}\frac{\partial^2 u_x}{\partial y^2} = 0 \tag{3a}$$

$$\left(\frac{E_1E_2}{E_1-\mu_{12}^2E_2}\right)\frac{\partial^2 u_y}{\partial y^2} + \left(\frac{\mu_{12}E_1E_2}{E_1-\mu_{12}^2E_2}+G_{12}\right)\frac{\partial^2 u_x}{\partial x\,\partial y} + G_{12}\frac{\partial^2 u_y}{\partial x^2} = 0 \tag{3b}$$

In the present study, a new potential function $\psi\,(x,y)$ is defined in terms of the two displacement components as follows:

$$u_x = \frac{\partial^2\psi}{\partial x\,\partial y} \tag{4a}$$

$$u_y = -\frac{1}{Z_{11}}\left[E_1^2\frac{\partial^2\psi}{\partial x^2} + G_{12}\left(E_1-\mu_{12}^2E_2\right)\frac{\partial^2\psi}{\partial y^2}\right] \tag{4b}$$

where $Z_{11} = \mu_{12}E_1E_2 + G_{12}\left(E_1-\mu_{12}^2E_2\right)$

With the above definition of ψ (x, y), the first equilibrium equation (3a) is automatically satisfied. Therefore, ψ has to satisfy the second equilibrium equation (3b) only. Expressing Eq. (3b) in terms of the potential function ψ, the condition that ψ has to satisfy is

$$E_1G_{12}\frac{\partial^4\psi}{\partial x^4} + E_2(E_1-2\mu_{12}G_{12})\frac{\partial^4\psi}{\partial x^2\partial y^2} + E_2G_{12}\frac{\partial^4\psi}{\partial y^4} = 0 \tag{5}$$

4. SOLUTION OF THE PROBLEM

For the model shown in Fig.1, the stiffened cantilever is considered to be of unit thickness and the potential function ψ is assumed to be

$$\psi = \sum_{m=1}^{\infty} X_m \cos\alpha y + Bx^3 \tag{6}$$

where X_m is a function of x only and $\alpha = m\pi/a$. Thus, X_m has to satisfy the ordinary differential equation

$$X_m'''' + \left(\frac{E_2}{G_{12}} - \frac{2\mu_{12}E_2}{E_1}\right)\alpha^2X_m'' + \frac{E_2}{E_1}\alpha^4X_m = 0 \tag{7}$$

where the ($'$) indicates differentiation with respect to x. The general solution of this differential equation can be given by:

$$X_m = A_me^{r_1x} + B_me^{r_2x} + C_me^{r_3x} + D_me^{r_4x} \tag{8}$$

$$r_1, r_2 = \alpha \left[\frac{1}{2} \left\{ \left(\frac{E_2}{G_{12}} - \frac{2\mu_{12} E_2}{E_1} \right) \pm \sqrt{\left(\frac{E_2}{G_{12}} - \frac{2\mu_{12} E_2}{E_1} \right)^2 - \frac{4 E_2}{E_1}} \right\} \right]^{1/2}$$

$$r_3, r_4 = -\alpha \left[\frac{1}{2} \left\{ \left(\frac{E_2}{G_{12}} - \frac{2\mu_{12} E_2}{E_1} \right) \pm \sqrt{\left(\frac{E_2}{G_{12}} - \frac{2\mu_{12} E_2}{E_1} \right)^2 - \frac{4 E_2}{E_1}} \right\} \right]^{1/2}$$

Here A_m, B_m, C_m, and D_m are constants. Now combining Eqs.(2), (4), (6), and (8), the expressions of stress and displacement components are obtained as follows:

$$u_x(x,y) = -\sum_{m=1}^{\infty} \alpha X'_m \sin\alpha y \tag{9}$$

$$u_y(x,y) = \frac{1}{Z_{11}} \sum_{m=1}^{\infty} \left[-E_1^2 X'''_m + G_{12}(E_1 - \mu_{12}^2 E_2) X_m \alpha^2 \right] \cos\alpha y - \frac{6BE_1^2}{Z_{11}} x \tag{10}$$

$$\sigma_{xx}(x,y) = -\frac{E_1 G_{12}}{Z_{11}} \sum_{m=1}^{\infty} \left[\alpha E_1 X''_m + \alpha^3 E_2 \mu_{12} X_m \right] \sin\alpha y \tag{11}$$

$$\sigma_{yy}(x,y) = -\frac{E_1 G_{12}}{Z_{11}} \sum_{m=1}^{\infty} \left[G_{12}\alpha^3 X_m + \alpha(\mu_{12} G_{12} - E_1) X''_m \right] \sin\alpha y \tag{12}$$

$$\sigma_{xy}(x,y) = -\frac{E_1 G_{12}}{Z_{11}} \sum_{m=1}^{\infty} \left[E_1 X'''_m + \alpha^2 \mu_{12} E_2 X'_m \right] \cos\alpha y - \frac{6BE_1^2 G_{12}}{Z_{11}} \tag{13}$$

For the present problem, it is seen that the boundary conditions on stiffened edges

$$u_x = 0 \quad \text{at} \quad y = 0 \text{ and } y = a$$
$$\sigma_{yy} = 0 \quad \text{at} \quad y = 0 \text{ and } y = a \tag{14}$$

are satisfied automatically.
The boundary conditions at the fixed edge, x=0, are

$$u_x(0, y) = 0 \quad \text{and} \quad u_y(0, y) = 0 \tag{15}$$

Now, the parabolic shear loading on the right lateral boundary of the beam, x=b, can be expressed mathematically as follows:

$$\sigma_{xy}(b,y) = -\frac{4P}{a^2}(y^2 - ay) = I_0 + \sum_{m=1}^{\infty} I_m \cos\alpha y \tag{16(a)}$$

The normal stress at this boundary is

$$\sigma_{xx}(b,y) = 0 \tag{16(b)}$$

where P is the maximum shear stress at y=a / 2. From Fourier integral formula, it can be written that

$$I_0 = \frac{2P}{3} \tag{17(a)}$$

and

$$I_m = -\frac{16P}{m^2\pi^2} \text{ for } m=2, 4, 6, \ldots, \infty \tag{17(b)}$$

By applying the associated boundary conditions in relevant equations, we get the following four equations in terms of the four unknowns A_m, B_m, C_m, and D_m.

$$r_1 A_m + r_2 B_m + r_3 C_m + r_4 D_m = 0 \tag{18(a)}$$

$$\left(\frac{Z_{12}}{Z_{11}}\alpha^2 - \frac{E_1^2}{Z_{11}} r_1^2 \right) A_m + \left(\frac{Z_{12}}{Z_{11}}\alpha^2 - \frac{E_1^2}{Z_{11}} r_2^2 \right) B_m + \left(\frac{Z_{12}}{Z_{11}}\alpha^2 - \frac{E_1^2}{Z_{11}} r_3^2 \right) C_m + \left(\frac{Z_{12}}{Z_{11}}\alpha^2 - \frac{E_1^2}{Z_{11}} r_4^2 \right) D_m = 0 \tag{18(b)}$$

$$\left(-\frac{E_1^2 G_{12}\alpha r_1^2}{Z_{11}} e^{r_1 b} - \frac{E_1 E_2 G_{12}\mu_{12}\alpha^3}{Z_{11}} e^{r_1 b} \right) A_m + \left(-\frac{E_1^2 G_{12}\alpha r_2^2}{Z_{11}} e^{r_2 b} - \frac{E_1 E_2 G_{12}\mu_{12}\alpha^3}{Z_{11}} e^{r_2 b} \right) B_m + \left(-\frac{E_1^2 G_{12}\alpha r_3^2}{Z_{11}} e^{r_3 b} - \frac{E_1 E_2 G_{12}\mu_{12}\alpha^3}{Z_{11}} e^{r_3 b} \right) C_m + \left(-\frac{E_1^2 G_{12}\alpha r_4^2}{Z_{11}} e^{r_4 b} - \frac{E_1 E_2 G_{12}\mu_{12}\alpha^3}{Z_{11}} e^{r_4 b} \right) D_m = 0 \tag{18(c)}$$

$$\left(\begin{array}{l}-\dfrac{E_1^2 G_{12} r_1^3}{Z_{11}} e^{r_1 b}-\\ \dfrac{E_1 E_2 G_{12}\mu_{12}\alpha^2 r_1}{Z_{11}} e^{r_1 b}\end{array}\right) A_m+\left(\begin{array}{l}-\dfrac{E_1^2 G_{12} r_2^3}{Z_{11}} e^{r_2 b}-\\ \dfrac{E_1 E_2 G_{12}\mu_{12}\alpha^2 r_2}{Z_{11}} e^{r_2 b}\end{array}\right) B_m+\left(\begin{array}{l}-\dfrac{E_1^2 G_{12} r_3^3}{Z_{11}} e^{r_3 b}-\\ \dfrac{E_1 E_2 G_{12}\mu_{12}\alpha^2 r_3}{Z_{11}} e^{r_3 b}\end{array}\right) C_m+\left(\begin{array}{l}-\dfrac{E_1^2 G_{12} r_4^3}{Z_{11}} e^{r_4 b}-\\ \dfrac{E_1 E_2 G_{12}\mu_{12}\alpha^2 r_4}{Z_{11}} e^{r_4 b}\end{array}\right) D_m = I_m \qquad 18(d)$$

where

$$Z_{12} = G_{12}\left(E_1-\mu_{12}^2 E_2\right)$$

Also, we get, $B=\dfrac{PZ_{11}}{9E_1^2 G_{12}}$.

The above four simultaneous algebraic equations 18(a)-18(d) can further be realized in a simplified form for the solution of the unknowns as follows:

$$\begin{bmatrix} r_1 & r_2 & r_3 & r_4 \\ P_1 & P_2 & P_3 & P_4 \\ Q_1 & Q_2 & Q_3 & Q_4 \\ R_1 & R_2 & R_3 & R_4 \end{bmatrix} \begin{bmatrix} A_m \\ B_m \\ C_m \\ D_m \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ I_m \\ 0 \end{bmatrix} \qquad (19)$$

where

$$\left.\begin{array}{l} P_i = \dfrac{Z_{12}}{Z_{11}}\alpha^2 - \dfrac{E_1^2}{Z_{11}} r_i^2 \\ Q_i = \left(\begin{array}{l} -\dfrac{E_1^2 G_{12} r_i^3}{Z_{11}} e^{r_i b} - \\ \dfrac{E_1 E_2 G_{12}\mu_{12}\alpha^2 r_i}{Z_{11}} e^{r_i b}\end{array}\right) \\ R_i = -\dfrac{E_1^2 G_{12}\alpha r_i^2}{Z_{11}} e^{r_i b} - \dfrac{E_1 E_2 G_{12}\mu_{12}\alpha^3}{Z_{11}} e^{r_i b} \end{array}\right] \quad i = 1, 2, 3, 4$$

5. RESULTS AND DISCUSSION

In this section, numerical results are presented for a boron / epoxy unidirectional composite cantilever beam. The effective mechanical properties of the boron/epoxy composite are E_1=28.29x10^4 MPa, E_2=2.415x10^4 MPa, μ_{12}=0.27 and G_{12}= 1.035x10^4 MPa. Furthermore, the aspect ratio of the cantilever beam used in obtaining the results is taken as b/a=3.0.

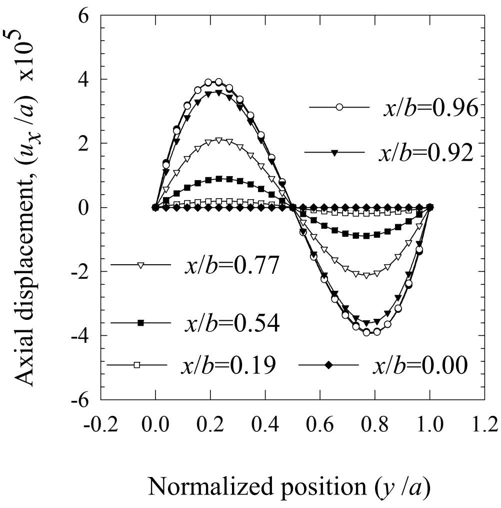

Fig 2(a). Distribution of normalized displacement component (u_x/a) at different sections of the beam

The variation of the normalized axial displacement component with y is shown in Fig. 2(a). At the section $0 \le y/a \le 0.5$, the axial displacement is positive and at the section $0.5 \le y/a \le 1.0$, this displacement is negative. The axial displacement increases with the increase of the value of x/b.

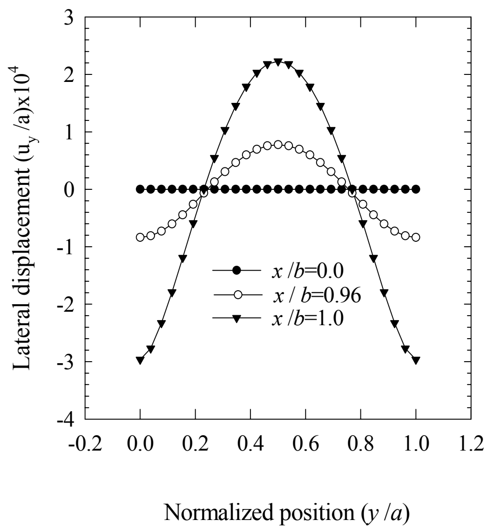

Fig 2(b). Distribution of normalized lateral displacement component (u_y/a) at different sections of the beam

Figure 2(b) illustrates the variation of normalized lateral displacement component with y at different sections of the cantilever. For higher value of x/b i.e. near the right lateral edge, the lateral displacement varies parabolically with y showing peak at y/a =0.5. As the ratio x/b decreases i.e. as the fixed support is approached, the parabolic shape gradually becomes flat with zero magnitude at x/b = 0.

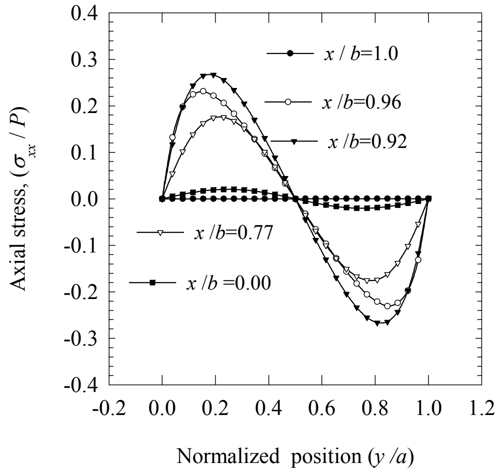

Fig 3(a). Distribution of normalized axial stress component (σ_{xx} / P) at different sections of the beam

Figure 3(a) illustrates the distribution of normalized axial stress component with the variation of *y* at different sections of the beam. It is observed that the stress distribution is antisymmetric with respect to *y*. The antisymmetric variation is significant in region between the two lateral edges $x/b > 0$ and $x/b < 1.0$. At the two edges, the variation of the stress with *y* is not so significant. At x/b=1.0, the stress is almost zero, which satisfies the physical boundary conditions.

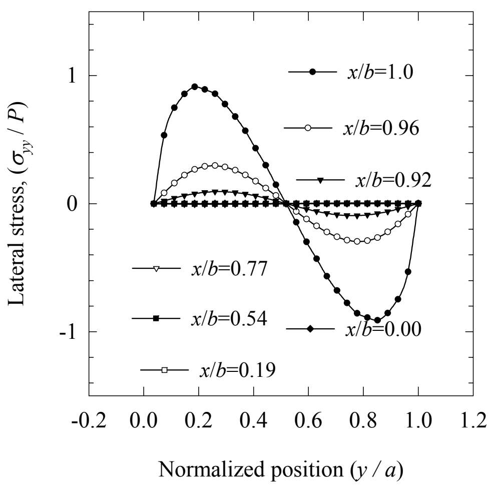

Fig 3(b). Distribution of normalized lateral stress components (σ_{yy} / P) at different sections of the beam

Figure 3(b) is the distribution of normalized lateral stress vs. *y* at different sections of the beam. This stress distribution is also antisymmetric with respect to *y*. However, in this case, the antisymmetric variation of the stress is significant at and near the right lateral edge. As the ratio of x/b decreases, the variation of antisymmetric stress distribution becomes more and more insignificant. Further, it is noted that at the two stiffened edges (y/a=0 and y/a=1.0), the lateral stress is zero, which satisfies the physical boundary conditions.

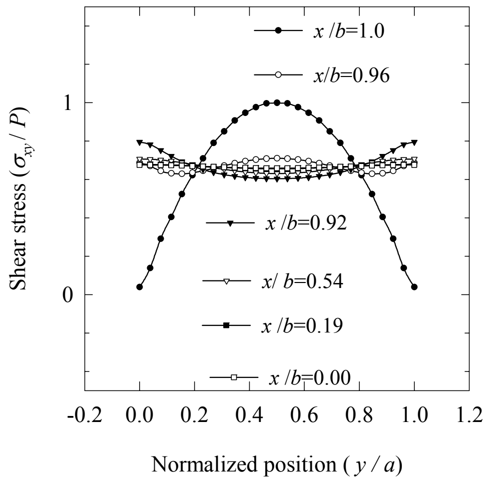

Fig 3(c). Distribution of normalized shear stress componer (σ_{xy} / P) at different sections of the beam

The distribution of normalized shearing stress as a function of *x* and *y* is shown in Fig.3(c). At x/b=1.0, i.e.at the right lateral edge, the shearing stress approaches the value of the applied load and thus satisfies the boundary conditions. The distribution of shearing stress is symmetric with respect to *y*. It is noted that the shearing stress is not zero at the stiffened edges for $x/b < 1.0$.

6. CONCLUSIONS

A new displacement potential approach has been used to analyze the states of stresses and displacements in a deep stiffened cantilever beam of composite material with mixed boundary conditions. No appropriate analytical approach was available in the literature which could satisfactorily provide the explicit information about the actual stresses at the critical regions of supports and loadings. Both the qualitative and quantitative results of the present stiffened cantilever beam problem of orthotropic composite materials establish the soundness as well as appropriateness of the present single function approach. The distinguishing feature of the present ψ-formulation over the existing approaches is that, here, all modes of boundary conditions can be satisfied exactly, whether they are specified in terms of loading or physical restraints or any combination of them; and thus the solutions obtained are promising and satisfactory for the entire regions of interest. From the analysis, it is clear that right vertical section is critical for normal stress in the *y*-direction because, at this section, the maximum value of this stress is nearly equal to the maximum value of the applied shearing stress.

7. REFERENCES

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8. NOMENCLATURE

Symbol	Meaning	Unit
$\psi(x,y)$	Displacement potential functio	
E_1	Elastic modulus in longitudinal direction	MPa
E_2	Elastic modulus in transverse direction	MPa
G_{12}	Shear modulus	MPa
μ_{12}	Major Poison's ratio	
μ_{21}	Minor Poison's ratio	
σ_{xx}	Axial stress	kPa
σ_{yy}	Lateral stress	kPa
σ_{xy}	Shear stress	kPa
P	Maximum value of shear stress	kPa
I_0, I_m	Constants	
A_m, B_m, C_m, D_m, B	Constants	
u_x	Axial displacement	m
u_y	Lateral displacement	m
a	Width of the cantilever beam	m
b	Length of the cantilever beam	m