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EXACT BENDING RELATIONSHIP BETWEEN HIGHER ORDER LEVINSON AND THIN PLATE SOLUTIONS FOR AN AXISYMMETRIC ANNULAR PLATE WITH LINEARLY VARYING THICKNESS

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ABSTRACT
Quite different seems to be the situation in the classical thin plate theory in which the influence of transverse shearing strains are neglected and for the majority of technical applications this classical plates theory gives sufficiently accurate result. However, there exist some important problems even in the technical application in which the results obtained from the classical plate theory are not acceptable. The effect of transverse shear deformation is pronounced in the bending analysis of plate. The objective of this paper is to establish the exact bending solutions of higher order shear deformable Levinson annular plate with linearly varying thickness using Kirchhoff solutions.

Keywords: Axisymmetric annular plates, Levinson, Exact bending solutions.

1. INTRODUCTION
The elastic circular plate with an annulus of non-uniform thickness is commonly used now-a-days for designing of machine parts, such as diaphragms of steam turbines, pistons of reciprocating engine etc. The analysis of the bending solution of a symmetrically loaded annular plate of variable thickness has been studied by many researchers [2–4], and also available in many textbooks on plate and shells such as Timoshenko and Woinowsky-Krieger (1959). But, most of the analyses are based on Kirchhoff thin plate theory, where the effects of transverse shear deformation are not considered. Recently, Wang and his co-researchers [5–7] have presented the exact bending relationship of first order shear deformable plate in terms of Kirchhoff solutions. Wang et. al [8] also introduced the bending relationship between the Kirchhoff and of higher order Levinson plate theory for a circular plate with constant thickness. The bending relationship between higher order Levinson plate and Kirchhoff thin plate is formulated with the help of mathematical similarities of the governing equations on the basis of load equivalence.

Deflections and radial stress are compared with the results obtained by Kirchhoff classical thin plate theory [1]. Also, the results for the maximum deflection and radial stress obtained from the present solution for the plate with external edge free, and internal edge clamped and supported are compared with the results obtained by Wang (1997) [6] (based on Mindlin plate theory) and Conway [4] (based on Kirchhoff thin plate theory). A perfect correlation has been observed.

2. GOVERNING EQUATIONS
The equations of equilibrium for a circular plate (Timoshenko & Woinowsky-Krieger 1959) are given by

$$(rM_r),_r - M_\theta = rQ_r \quad \text{and} \quad (rQ_r),_r = -rq \tag{1a, b}$$

According to Kirchhoff, the relation between bending moment and displacements are given by

$$M_r^K = -D\left(w,_{rr}^K + \frac{\nu}{r} w,_{rr}^K\right), \text{ and}$$
$$M_\theta^K = -D\left(\nu w_{,rr}^K + \frac{1}{r} w_{,r}^K\right) \tag{2a, b}$$

Equations (2a), (2b), and (1a) yield,

$$-D\left[\left(w_{,rrr}^K + \frac{1}{r} w_{,rr}^K - \frac{1}{r^2} w_{,r}^K\right) + \frac{3}{r}\left(w_{,rr}^K + \frac{\nu}{r} w_{,r}^K\right)\right] = Q_r^K \tag{3}$$

The Levinson stress-resultant and displacement relations [8] are given by

$$M_r^L = \frac{D}{5}\left[4\left(\psi_{,r}^L + \frac{\nu}{r}\psi^L\right) - \left(w_{,rr}^L + \frac{\nu}{r} w_{,r}\right)\right] \tag{4a}$$

$$M_\theta = \frac{D}{5}\left[4\left(\nu\psi,_r^L + \frac{1}{r}\psi^L\right) - \left(\nu w_{,rr}^L + \frac{1}{r} w_{,r}\right)\right] \tag{4b}$$

$$Q_r^L = \frac{2}{3} G\alpha r \left(\psi + w,_r^L\right) \tag{4c}$$

where, □ = *h*/*r* = *H*/*a* is the thickness ratio of the plate. Equations (4a), (4b), and (1a) yield,

$$\frac{D}{5}\left[4\left(\psi_{,rr}^L + \frac{1}{r}\psi_{,r}^L - \frac{\psi^L}{r^2}\right) - \left(w_{,rrr}^L + \frac{1}{r}w_{,rr}^L - \frac{1}{r^2}w_{,r}^L\right) + \frac{3}{r}\left\{4\left(w_{,r}^L + \frac{\nu}{r}\psi\right) - \left(w_{,rr}^L + \frac{\nu}{r}w_{,r}^L\right)\right\}\right] = Q_r^L \tag{5}$$

Based on the concept of load equivalence of two theories, equation (1b) gives,

$$\left(rQ_r^L\right)_{,r} = \left(rQ_r^K\right)_{,r} = -rq \tag{6}$$

Using the regularity condition at $r = 0$ for circular plate or the statical boundary condition $Q_r = 0$ at the free edge for the annular plates, it follows from equation (6) the shear force of Levinson and the corresponding Kirchhoff plate is same, *i.e*,

$$Q_r^L = Q_r^K \tag{7}$$

For ν = 1/3, equations (3), (5), and (7) yield

$$r^2 y_{,r} + 4 r y_{,r} = 0, \text{ where, } y = \frac{4}{5}\psi^L - \frac{1}{5}\psi_{,r}^L + w_{,r}^K \tag{8}$$

The solution of equation (8) is

$$y = \frac{4}{5}\psi^L - \frac{1}{5}w_{,r}^L + w_{,r}^K = A + \frac{B}{r^3} \tag{9}$$

where, *A* and *B* are constants of integration.

To establish the relation of stress-resultant between Levinson and Kirchhoff plates, multiply equation (9) by (ν/*r*) and add with the differential form of equation (9) w.r.t. '*r*'. The relation becomes as follows,

$$\frac{M_r^L}{D} = \frac{M_r^K}{D} + \frac{\nu}{r}A + \frac{B}{r^4}(\nu - 3) \tag{10}$$

3. PLATE WITH A UNIFORMLY DISTRIBUTED LOAD OF INTENSITY *'q'*

By integrating equation (1b), one can obtain the following relationship with the help of equation (4c) and equation (7),

$$\frac{\psi^L}{5} + \frac{1}{5}w_{,r}^L = \frac{3}{10\,G\alpha r}Q_r^L = \frac{3\,q}{20\,G\alpha}\left(\frac{b^2}{r^2} - 1\right) \tag{11}$$

Adding equation, (9) and (11) to eliminate the slope of Levinson, and obtain the following relationship between the rotations of Levinson plate with the slope of Kirchhoff plates,

$$\psi^L + w_{,r}^K = A + \frac{B}{r^3} + \frac{3\,q}{20\,G\alpha}\left(\frac{b^2}{r^2} - 1\right) \tag{12}$$

Use equation (11) and (9) to eliminate the rotation (ψ^L) of Levinson plate and it yield,

$$w_{,r}^L = w_{,r}^K - A - \frac{B}{r^3} + \frac{3\,q}{5\,G\alpha}\left(\frac{b^2}{r^2} - 1\right) \tag{13}$$

Integrating equation (13) to establish the relationship between the deflection of Levinson and Kirchhoff plates as follows,

$$w^L = w^K - A\,r + \frac{B}{2\,r^2} - \frac{3\,q\,r}{5\,G\alpha}\left(1 + \frac{b^2}{r^2}\right) + C \tag{14}$$

Three cases of some practical importance are considered as follows:

3.1 External Edge Clamped and Supported, Internal Edge Clamped (ECS – IC)

The boundary conditions are,

$$\left.\begin{array}{ll} r = a, & \psi^L = w_{,r}^K = 0 \\ r = b, & \psi^L = w_{,r}^K = 0 \end{array}\right\} \tag{15}$$

The constants *A* and *B* are found from equation (12),

$$A = \frac{2\,q}{5\,E\alpha}\frac{n\,(n^2-1)}{(n^3-1)}, \text{and} \quad B = -\frac{2\,q}{5\,E\alpha}\frac{a^3(n^2-1)}{n^2(n^3-1)} \tag{16}$$

where, *n* =*a*/*b* is annular ratio of the plate.

The constant *C* of the equation (14) is obtained from the condition of zero deflection at the outer edge (*i.e*., at $r = a$, $w^K = w^L = 0$). Evaluating *A*, *B*, and *C*, the deflection equation is to be found,

$$\overline{w}^L = \overline{w}^K + \frac{1}{5}(\beta - 1)\,\alpha^2[(n+1)(1 + 9\beta - 10n^3\beta^2) - 8n^2(\beta-1)\beta]/(n^2 + n + 1)n^2\beta^2 \tag{17}$$

with, $\overline{w}^L = \dfrac{w^L E H^3}{q\,a^4}$,and $\overline{w}^K = \dfrac{w^K E H^3}{q\,a^4}$ are the non-dimensional deflection of Levinson and Kirchhoff plates, and . represent the nondimensional radius (*r*/*a*).

The relation of radial stress between Levinson and Kirchhoff are obtained from equation (10) and (16),

$$\overline{\sigma}^L = \overline{\sigma}^K + \alpha^2\frac{3(n^2-1)}{40(n^3-1)}\left(n + \frac{8}{n^2\beta^3}\right) \tag{18}$$

with, $\overline{\sigma}^L = \dfrac{\sigma_r^L}{q}\left(\dfrac{H}{a}\right)^2$, $\overline{\sigma}^K = \dfrac{\sigma_r^K}{q}\left(\dfrac{H}{a}\right)^2$, and

$$\sigma_r^K = \frac{6M_r^K}{h^2} \tag{19}$$

3.2 External Edge Simply Supported and Internal Edge Clamped (ESS – IC)

The boundary conditions are,

$$\left.\begin{array}{ll} r = a, & M_r^L = M_r^K = 0 \\ r = b, & \psi_r^L = w_{,r}^K = 0 \end{array}\right\} \tag{20}$$

Putting above boundary condition of equation (20) into equation (10) and (12), constants A and B are found,

$$A = B = 0 \tag{21}$$

The deflection of Levinson plate in terms of Kirchhoff plate is obtained from equation (14),

$$w^L = w^K - \frac{8\,q\,r}{5\,E\alpha}\left(1 + \frac{b^2}{r^2}\right) + C \tag{22}$$

Constant C is to be found from the condition of zero deflection at the outer edge (*i.e.,* at $r = a$, $w^K = w^L = 0$) and the equation (22) yields in nondimensional form as,

$$\overline{w}^L = \overline{w}^K + \frac{8}{5}\alpha^2\left\{(1-\beta)\left(1 - \frac{1}{n^2\beta}\right)\right\} \tag{23}$$

The relationship of radial stresses between Levinson and Kirchhoff are obtained from equation (10) and (21) and given same value.
The maximum stress are found at inner edge [1],

$$\max\overline{\sigma}_r^L = \max\overline{\sigma}_r^K = 3\left[\frac{n^3}{n^3+8}\ln(n) + \frac{(n-1)(11n^2+13n-32)}{6(n^3+8)}\right] \tag{24}$$

3.3 External Edge Free, Internal Edge Clamped and Supported (EF – ICS)

The boundary conditions are,

$$\left.\begin{array}{ll} r = a, & M_r^L = M_r^K = 0 \\ r = b, & \psi_r^L = w_{,r}^K = 0 \end{array}\right\} \tag{25}$$

The shear force,

$$Q_r = \frac{q}{2}\left(\frac{a^2}{r} - r\right) \tag{26}$$

Constant A and B are to be obtained,

$$A = -\frac{16\,q\,(n^2-1)}{5\,E\,\alpha\,(8+n^3)}\,; \quad B = -\frac{2\,q\,(n^2-1)\,a^3}{5\,E\,\alpha\,(8+n^3)} \tag{27}$$

The deflection equation of Levinson plate in terms of Kirchhoff plate is obtained from equation (14) with condition of zero deflection at inner edge (*i.e.* at $r = b$, $w^K = w^L = 0$). The maximum deflection occurred at the free edge (*i.e.* at $r = a$)

$$\overline{w}^L = \overline{w}^K + \frac{1}{5}\frac{(n^2-1)\alpha^2}{(8+n^3)}\left[\left\{16\left(\beta - \frac{1}{n}\right) + \left(n^2 - \frac{1}{\beta^2}\right)\right\} + \frac{8}{5}\alpha^2\left\{\frac{(1+n)^2}{n} - \beta\left(1 + \frac{1}{\beta}\right)\right\}\right] \tag{28}$$

The Levinson radial stresses in terms of Kirchhoff plate are to be found from equation (10) and (27),

$$\overline{\sigma}_r^L = \overline{\sigma}_r^K + \alpha^2\,\frac{3\,(n^2-1)}{5\,(n^3+8)}\left(\frac{1}{\beta^3} - 1\right) \tag{29}$$

Whereas, Wang [6] expressed that maximum radial stress of Mindlin plate is same of Kirchhoff plate.

4. PLATE WITH A TOTAL LOAD '*P*' UNIFORMLY DISTRIBUTED AROUND A CENTRAL HOLE

The shear force for the plate with total load P uniformly distributed around a central hole yield,

$$Q_r^L = Q_r^K = -\frac{P}{2\,\pi\,r} \tag{30}$$

Equation (30) and equation (4c) yield,

$$\frac{\psi^L}{5} + \frac{1}{5}w_{,r}^L = \frac{3}{10\,G\alpha r}Q_r^L = -\frac{2\,P}{5E\pi\alpha r^2} \tag{31}$$

The relationship between the rotations of Levinson plate and the slope of Kirchhoff plates are obtained by adding equation (9) with equation (31),

$$\psi^L + w_{,r}^K = A + \frac{B}{r^3} - \frac{2\,P}{5E\pi\alpha r^2} \tag{32}$$

Again, equation (31) and equation (9) yield,

$$w_{,r}^L = w_{,r}^K - A - \frac{B}{r^3} - \frac{8\,P}{5E\pi\alpha r^2} \tag{33}$$

The deflection of Levinson plate in terms of Kirchhoff plate to be obtained from equation (33),

$$w^L = w^K - A\,r + \frac{B}{2\,r^2} - \frac{3\,P}{5\,\pi G\alpha\,r} + C \tag{34}$$

4.1 Plate with External Edge Simply Supported, and Free at Inner Edge (ESS – IF)

The boundary conditions are

$$\left.\begin{array}{ll} r = a, & M_r^L = M_r^K = 0 \\ r = b, & M_r^L = M_r^K = 0 \end{array}\right\} \tag{35}$$

Constants A and B of equation (34) are to be found by putting boundary conditions into equation (10),

$A = B = 0,$ (36)

The non-dimensional deflection of Levinson plate in terms of Kirchhoff plate are to be found from equation (34) after determining the constant *C* from the condition of zero deflection at external edge,

$$\overline{w}^L = \overline{w}^K + \frac{8}{5}\alpha^2\left\{(1-\beta)\left(1-\frac{1}{n^2\beta}\right)\right\}, \quad (37)$$

with $\overline{w}^L = \frac{w^L EH^3}{P\,a^2}$, and $\overline{w}^K = \frac{w^K EH^3}{P\,a^2}$

The value of radial stresses for the Levinson plate and Kirchhoff plate are found to be same from the equation (10) and (36).

4.2 External Edge Clamped and Supported, Internal Edge Clamped (ECS – IC)

The boundary conditions are,

$$\left.\begin{aligned} r &= a, & \psi^L &= w^k_{,r} = 0 \\ r &= b, & \psi^L &= w^K_{,r} = 0 \end{aligned}\right\} \quad (38)$$

Constants *A* and *B* are to be obtained from equation (32),

$$A = \frac{2P\,n^2(1-n)}{5\pi E\alpha a^2(1-n^3)}, \qquad B = \frac{2Pa\,(1-n^2)}{5\pi E\alpha\,(1-n^3)} \quad (39)$$

The deflection of Levinson plate in terms of Kirchhoff plate are obtain from the equation (34) with the condition of zero deflection at outer edge,

$$\overline{w}^L = \overline{w}^K + \frac{2}{5\pi}\alpha^2\left[\frac{1}{(1-n^3)}\left\{\frac{n^2(1-n)(1-\beta)}{1} + \frac{1}{2}(1-n^2)\left(\frac{1}{\beta^2}-1\right)\right\} + 4\left(\frac{1}{\beta}-1\right)\right] \quad (40)$$

The relationship between Levinson and Kirchhoff plate for radial stresses yield from equation (10)and (39),

$$\overline{\sigma}^L_r = \overline{\sigma}^K_r + \frac{3\alpha^2}{40\pi(1-n^3)}\left[n^2(1-n) - \frac{8(1-n^2)}{\beta^3}\right] \quad (41)$$

with $\overline{\sigma}^L_r = \frac{\sigma^L_r H^2}{P}$, and $\overline{\sigma}^K_r = \frac{\sigma^K_r H^2}{P}$

4.3 External Edge Simply supported, internal edge clamped (ESS – IC)

The boundary conditions are,

$$\left.\begin{aligned} r &= a, & M^L_r &= M^K_r = 0 \\ r &= b, & \psi^L_r &= w^K_{,r} = 0 \end{aligned}\right\} \quad (42)$$

The constants *A* and *B* are to be found by putting the boundary conditions of equation (42) into equation (10) and equation (32),

$$A = \frac{16\,P}{5\,\pi E\,\alpha b^2\,(8+n^3)}, \qquad B = \frac{2\,Pa^3}{5\,\pi E\,\alpha b^2\,(8+n^3)} \quad (43)$$

The deflection Levinson plate in terms Kirchhoff plate yields by putting the value of *A* and *B* into equation (34), and determining the constant *C* from the condition of zero deflection at the outer edge,

$$\overline{w}^L = \overline{w}^K + \frac{2}{5\pi}\alpha^2\left[\frac{3n^2}{(8+n^3)}\left\{\frac{1}{6}\left(\frac{1}{\beta^2}-1\right) + \frac{8}{3}(1-\beta)\right\} + 4\left(\frac{1}{\beta}-1\right)\right] \quad (44)$$

Using the equation (14) and (43), the radial stresses of Levinson plate are obtained in terms of Kirchhoff plate

$$\overline{\sigma}^L_r = \overline{\sigma}^K_r + \alpha^2\frac{3n^2}{5\pi(n^3+8)}\left[1-\frac{1}{\beta^3}\right] \quad (45)$$

5. RESULTS AND DISCUSSION

The Poisson's ratio for the bending analysis of linearly varying annular elastic plates are taken as $\nu = 1/3$. Results are presented for the plate under uniformly distributed load of intensity *q*, and total load *P* uniformly distributed around the central hole. Plate with ECS – IC, ESS – IC, EF – ICS, and ESS – IF are considered in the present study. Nondimensional deflection vs. nondimensional radial coordinate of higher order Levinson plate and Kirchhoff plate are plotted from Fig.1 to Fig.5. Table 1, and Table 2 shows the result for maximum deflection and stresses of various annular ratios for EF - ICS plate under uniformly distributed load. Results are compared with the result obtained from Wang (based on Mindlin plate theory and shear correction factor taken, $k^2 = 5/6$) [6] and Conway (based on Kirchhoff thin plate theory) [2]. Here it has been observed that the Levinson transverse deflection is greater than the deflection obtained from Mindlin and Kirchhoff theory and the differences are increased with the increasing the thickness ratio (*H/a*). Also, these differences are increased with the decrease of annular ratio ($n = a/b$). For $a/b = 1.25$ and $H/a = 0.2$, the present result for transverse deflection (based on Levinson) is 16.75% and 35.75% more than the result obtained by Wang and Conway respectively. Whereas, these differences are only 2.14% and 6.17% for $a/b = 5$ and $H/a = 0.2$. According to Wang [6], radial stresses obtained from Kirchhoff and Mindlin are same. But, the present results based on Levinson plate theory deviate for EF – ICS plate from 18.54% to 1.68% for different cases.

Fig. 1 and Fig. 2 shows ECS – IC plate with annular ratio $a/b = 10/3$. It has been observed that the Levinson transverse deflection increases with the increase of thickness ratio. For $H/a = 0.2$, the present result shows

16.25% and 16.41% more from Kirchhoff solution for maximum deflection of the plate with uniformly distributed load *q* and central load *P* respectively. Fig. 3 and Fig. 4 shows ESS – IC plate under load *q* and ESS – IF plate under load *P* respectively with annular ration *a/b* = 10/3. It has been observed that for both cases the higher order shear effect is negligible and in the present result the deflection increases from 1% to 2% for maximum deflection. Fig.5 represent ESS – IC plate under load *P*. Here present result shows the effect of higher order shear deformation on deflection and this effect is more prominent near to the clamped edge support. Most of the cases the Levinson stresses are either equal or very small differences with the stresses obtained from Kirchhoff plate theory.

6. CONCLUSIONS

It has been observed that present solution procedure is simple to solve the higher order Levinson plate theory without any mathematical difficulties and numerical calculation. Present solution gives a closer result of Kirchhoff solution and shows how the effect of transverse shear deformation influences for the deflection and radial stresses. The effect of transverse shear deformation is more for the case of clamped edge and increases with the increase of thickness ratio (*H/a*). This shear effect is also increases with the decrease of annular ratio (*a/b*). The accuracy for the nondimensional transverse deflection and radial stresses obtained by the present method is found to be fairly good in agreement for the different boundary conditions and loading.

Table 1: Comparison of nondimensional maximum deflection for EF–ICS plate under uniformly distributed load of intensity *q*

Annular Ratio *n = a/b*	Conway et.al [2] $\max.\overline{w}^K$	Wang [6] ^{M} $\max.\overline{w}^M$				Present result ^{L} $\max.\overline{w}^L$			
		H/a				H/a			
		0.05	0.1	0.15	0.2	0.05	0.1	0.15	0.2
1.25	0.00372	0.0041	0.0048	0.0060	0.0077	0.0121	0.0370	0.0786	0.1367
1.5	0.0453	0.0461	0.0485	0.0525	0.0587	0.0543	0.0814	0.1265	0.1897
2	0.4010	0.4032	0.4104	0.4224	0.4392	0.4119	0.445	0.5001	0.5773
3	2.1190	2.1264	2.1456	2.1776	2.2224	2.1355	2.1819	2.2592	2.3675
4	4.2450	4.2557	4.2881	4.3421	4.4177	4.2647	4.3241	4.4232	4.5619
5	6.2830	6.2981	6.3441	6.4209	6.5285	6.3068	6.3792	6.4998	6.6686

Table 2: Comparison of non-dimensional maximum radial stresses for EF–ICS plate under uniformly distributed load of intensity *q*

Annular Ratio *n = a/b*	Conway et.al [2] $\max.\overline{\sigma}_r^K$	Wang [6] ^{M} $\max.\overline{\sigma}_r^M = \max.\overline{\sigma}_r^K$	Present result ^{L} $\max.\overline{\sigma}_r^L$			
			H/a			
			0.05	0.1	0.15	0.2
1.25	0.249	0.249	0.2491	0.2493	0.2497	0.2953
1.5	0.638	0.638	0.6384	0.6396	0.6415	0.6443
2	3.96	3.96	3.962	3.9679	3.9777	3.9915
3	13.64	13.64	13.649	13.676	13.720	13.782
4	26.0	26.0	26.02	26.079	26.177	26.315
5	40.63	40.63	40.664	40.774	40.942	41.313

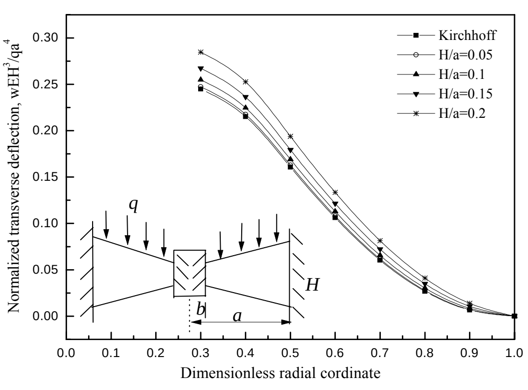

Fig 3. Deflection profiles of ECS-IC plates under uniformly distributed load of intensity *q*

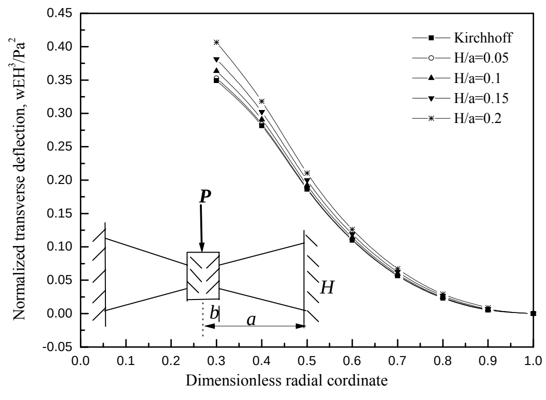

Fig 2. Deflection profiles of ECS-IC plates under load *P* uniformly distributed around a central hole

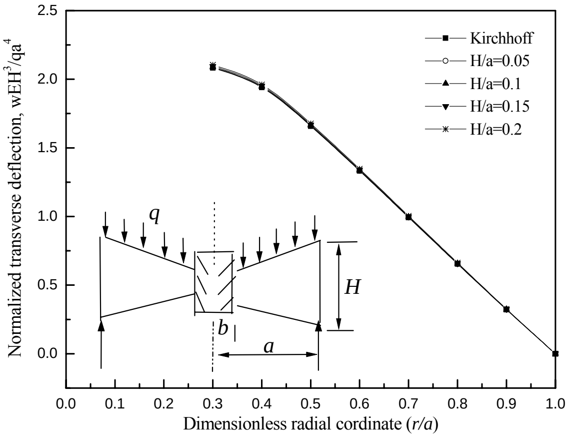

Fig 3. Deflection profiles of ESS-IC plates under uniformly distributed load of intensity

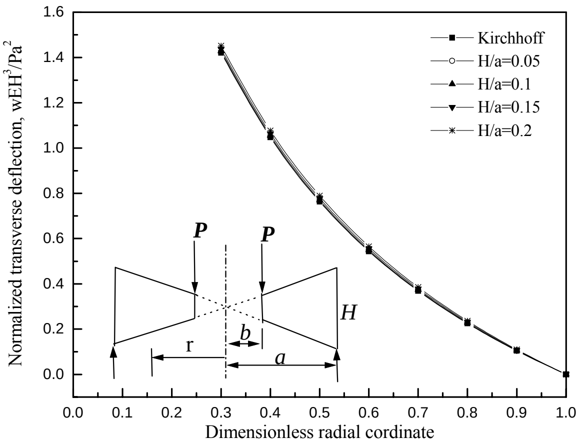

Fig 4. Deflection profiles of ESS-IF under load uniformly distributed around a central hole

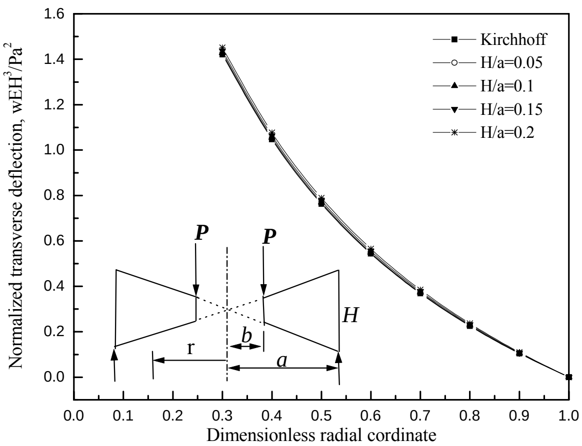

Fig 5. Deflection profiles of ESS-IC under load uniformly distrib`uted around a central hole

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8. NOMENCLATURE

Symbol	Meaning
M_r , M_θ	Radial and tangential Moment
Q_r	Shear force
D	Flexural rigidity of plate
β	Nondimensional radius (*r/a*)
n	Annular ratio (*a/b*)
α	Thickness ratio (*H/a* or *h/r*)
ψ^L	Rotation due to Levinson
$(\ )_{,r}$	Derivative w.r.t '*r*'
$\overline{w}$	Nondimensional deflection
$\overline{\sigma}$	Nondimensional stress