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A NEW APPROXIMATE SOLUTION FOR INITIAL DEVELOPMENT OF PLASTIC REGIONIN ELASTIC-PLASTIC TORSION OF SECTIONS WITH CONSTANT YIELD STRESS FUNCTION.

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ABSTRACT

In this paper elastic-plastic torsion of non circular solid sections was considered. A discontinuous stress distribution at the boundaries was assumed to simplify the differential equation of torsion. This conjecture seems irrational firstly, but the results shows consistency with finite element results in initial evolution of plastic region. In this way elastoplastic torsion of rectangular, L and H shape sections are fully analyzed. Finally the results are confirmed by the results of finite element method.

Keywords: Torsion, Ealstoplastic, constant yield stress

1.INTTODUCTION

For investigating Elastic plastic torsion of prismatic sections with constant yield stress, numerous efforts have been done before 1990, For example: Analytical solution to elastoplastic torsion of prismatic bars with constant yield stress has been extensively studied and concluded by late 1980's. Nadai [1] proposed a membrane analogy for the elastoplastic torsion of prismatic bars. In his work, a surface, corresponding to the stress function Ψ in the plastic region was assumed and a semi inverse solution for torsion of oval solid sections was given. elastic-plastic torsion is formulated variationally. On the other hand elastic-plastic problem for complex shear can be adequately investigated by analytical means. In a more complicated torsion problem, when the plastic zone becomes comparable with the size of the rod cross section, concluding logical solutions are rather scarce. First of all, Sokolovsky studied a rod of near-elliptical oval cross section [2]. The solution was obtained in 1942 by the semi-inverse method. Galin [3], using another semi-inverse method, solved several elastic-plastic problems for rods with a near-polygonal section (particularly, near-rectangular sections). Kachanov obtained a solution for a rod with square cross section using a variational method [4]. Southwdl and Takaci, using relaxation methods [5], solved the elastic-plastic problem for L-shaped, square, and triangular profiles as well as for a using the same methods. Using the technique of local variations, Banichuk, Petrov, and Chernousko figured out the elastic-plastic problem for a square section and also for a polygon of a particular shape [9]. Perlin licked the problem of an oval-shaped rod under conditions of partial enclosure of the elastic core by the plastic zone [10]. Annin and Sadovsky used the method of straight lines, and obtained an approximate solution for a rectangular section with the plastic zones developing near one pair of sides only [11]. In other works, Bland [12], in his article showed implementation of an algorithm for elastoplastic torsion. Arutyunyan and Radayev [13] devised a new method to workout Elastoplastic torsion of a cylindrical rod with finite deformations. Jinag and Henshall [14] found a coupling cross-section finite element model for torsional analysis of prismatic bars. Ryoo and Lee [15] worked out a computational verification method for determining existence of solution for elastoplastic torsion problems with uniqueness.

2.THEORY

For achieving to an approximate solution of elstoplastic torsion of certain sections without stress hardening, a simplifying assumption was supposed; i.e. presence of discontinuous stress function in the boundry between elastic and plastic zones. A variety of numerical and analytical solutions exist, but there is no general approach for the analytical solution.

Here the main idea of the approximate solution is removing the requisite continuity of stress in the boundary of elastic and plastic region. It may resemble unrealistic even though this is a wide spread problem which is utilized in finite element analysis. In 90% of cases only the displacement function is continuous.

Professionals have contrived some methods for evading from this problem, for example they use mesh less analysis. Therefore this assumption is not so illegal. The continuous function ψ is included here too (same as elliptical displacement function in finite element), but in initial considerations the initiative idea of membrane analogy for elastoplastic torsion of section is utilized. The assumption will arrive at decoupling of governing differential equations and breaking them to two partial differential equations in the field. The semi inverted solution of elastic torsion has been done before. By incorporation of hook's law; incorporating consistency equations and assuming a function that satisfies stress equations, governing partial differential equations will be:

$$\frac{\partial \psi(x,y)}{\partial s}\Big|_{\Gamma} = 0 \qquad (1)$$

$$\nabla^2 \psi(x,y) = -2G\theta \qquad (2)$$

ψ is a torsion function and G is shear modulus and θ is the torsional angle per unit length. For a general body ψ has a constant value on the boundry that will be zero for external periphery, but for internal geometries (holes …) it gains a non zero constant. The shear stresses τ_{xz} and τ_{zy} is as follows:

$$\tau_{xz} = \frac{\partial \psi(x,y)}{\partial y} \qquad (3)$$

$$\tau_{yz} = -\frac{\partial \psi(x,y)}{\partial x} \qquad (4)$$

In which Z axis is aligned along the length of the beam. The physical initiative of governing differential equation is similar to membrane analogy, that is one side is under atmospheric pressure, since governing equation is :

$$\nabla^2 Z(x,y) = -\frac{p}{s} \qquad (5)$$

In which Z is the height of membrane, P is the pressure of atmosphere and s is the constant of tension. Z is proportional to ψ , apparently; P to 2θ and s to $\frac{1}{G}$.

For plastic solution of sections differential equation of system is simply developed according to Mises yield criterion, with pure shear. We have:

$$|\sigma_1| = |\sigma_2| = |\tau| \qquad (6)$$

By substitution in misses equation, (6) becomes:

$$|\tau| = \frac{\sqrt{3}}{3} Y \qquad (7)$$

Now with respect to intensity of τ_{xz} and τ_{zy} we have:

$$\left(\frac{\partial \psi}{\partial x}\right)^2 + \left(\frac{\partial \psi}{\partial y}\right)^2 = K^2 \qquad (8)$$

Where k is the shearing yield stress computed by Eq. (7). Physical interpretation of ψ for fully plastic torsion of section is similar to a surface filled with sand, by other words it is similar to a surface, extruded under a certain angle which can be simply produced by software like solid works.

The initiative of elastoplastic torsion of sections without stress hardening can be explained by virtue of the two described situations; elastic and fully plastic torsion. As a section is subjected to torsion; according to membrane analogy, by increasing the torsional angle the section acts like a membrane which its pressure is raised until it reaches to the constant gradient surface (Physical yielding of metals). These constant gradient surfaces will constraint the growth in the perpendicular direction. In fact we can visualize the progress of plastic zone in the elastic border.

The next step for simplification and solving the differential equations approximately, is omitting the necessity of stress continuity; in the border of elastic and plastic zone. This will decouple the equations of the system. The coupled system equations are:

$$\psi(x,y)\big|_{\Gamma} = 0 \qquad (9)$$

$$\nabla^2 \psi(x,y) = -2G\theta \qquad (10)$$

$$\left(\frac{\partial \psi}{\partial x}\right)^2 + \left(\frac{\partial \psi}{\partial y}\right)^2 = S_y^{\,2} \qquad (11)$$

In which $\frac{\partial \psi}{\partial y}$ and $\frac{\partial \psi}{\partial x}$ are continuous functions. In addition $(\frac{\partial^2 \psi}{\partial y^2})$ and $(\frac{\partial^2 \psi}{\partial x^2})$ are bounded. In decoupled conditions we have:

$$\psi(x,y)\big|_{\Gamma} = 0 \qquad (12)$$

$$\nabla^2 \psi(x,y) = -2G\theta \qquad (13)$$

$$\left(\frac{\partial \psi}{\partial x}\right)^2 + \left(\frac{\partial \psi}{\partial y}\right)^2 = S_y^{\,2} \qquad (14)$$

In which ψ is a continuous function. If the derivative of ψ is not to be continuous the equations become decoupled. The upper bound of constant gradient will no longer prevent the membrane from growth, but the two surfaces will be determined independently for any desired torsional angle and the resulting intersecting volume is computed for the determination of required torsional moment of the section.

For an approximate elastoplaic solution of a certain section with a discontinuous stress function; the elastic and fully plastic solutions of the sections should be found. Then with computational software the minimum ψ will be determined at each point.

The exerted moment on the section is as follows:

$$T = 2 \int\int \psi (x, y) dxdy \qquad (15)$$

To do this software is developed by visual FORTRAN and visual basic in which a refined finite difference method is utilized. In this approach for border nodes; with respect to the distance between node and the border; the value of nodes are computed. Accordingly accuracy of computations will be raised. The finite difference equations are as follows:

$$\nabla^2\psi = \frac{2\psi_1}{\beta_1(1+\beta_1)} + \frac{2\psi_2}{\beta_2(1+\beta_2)} + \frac{2\psi_3}{(1+\beta_1)} + \frac{2\psi_4}{(1+\beta_2)} - \left[\frac{2}{\beta_1} + \frac{2}{\beta_2}\right]\psi_0 \qquad (16)$$

$$\beta_i = \frac{h_i}{h} \qquad (17)$$

Where h_i is the distance between node and the border while h is the distance between nodes. For developing the solving algorithm of a plastic section, the following statement is used; if the equation of section is analogous to a parametric curve as the following :

$$\Psi(x, y) = ku_\Gamma(x, y) \qquad (18)$$

In which $u_\Gamma(x, y)$ is the minimum distance of point to the boundary Γ , (k constant); then the equation will satisfy this:

$$(\frac{\partial\psi}{\partial x})^2 + (\frac{\partial\psi}{\partial y})^2 = K^2 \qquad (19)$$

After extracting ψ for elastic solution (which is dependent to θ) and then for its plastic counterpart, it is the time to compute ψ for elastoplastic condition. At this stage according to the desired θ angle, The value of ψ for elastic solution will be computed and its value will be compared with its plastic partner on each node; then the minimum of which will be assigned as the desired ψ . From this function shearing stresses can be computed by derivation and the moment by integrating. If the torsional angle is not definite and only the moment is known, then the software is capable of computing the torsional angle with a bisectional algorithm. In fig (1) the algorithm the angle finding was presented.

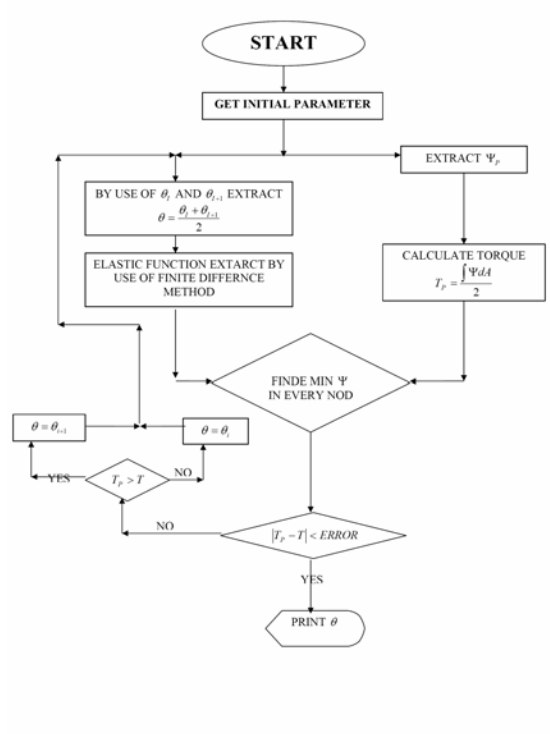

Fig 1: the torsional angle computing algorithm by halving Lastly it is worth noting that the software is a coupled program with visual basic and fortran.

4. RESULTS

Surface of Ψ function for full plastic torsion of H and L section shown in fig(2),(3).

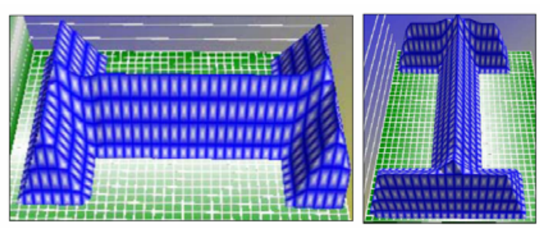

Fig 2: Surface of Ψ function for full plastic torsion of H section

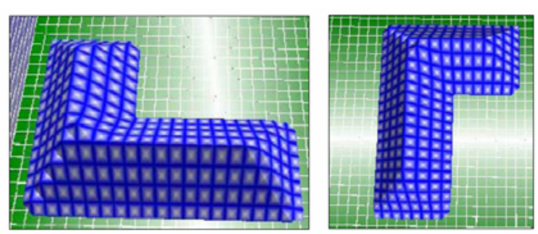

Fig 3: surface of Ψ function for full plastic torsion of L section

In the following the progress pattern of plastic zone and torsional angle curve via applied moment for rectangle, H and L shaped section is compared with FEM result.

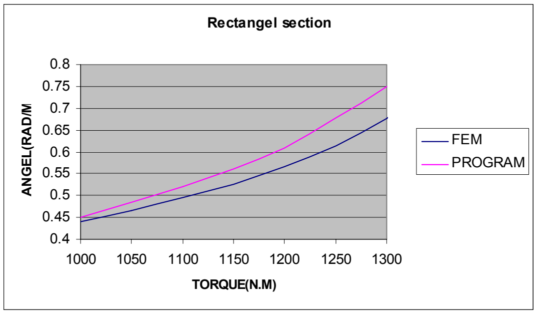

Fig 4: Torsional angle via applied moment for rectangular sections

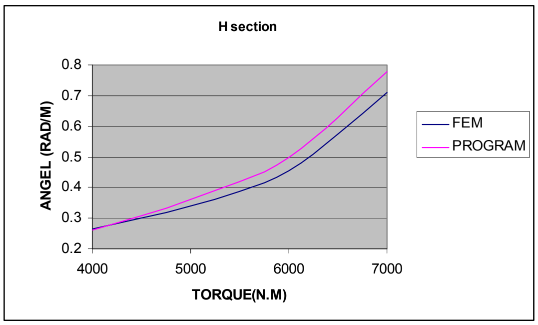

Fig 5: Torsional angle via applied moment for H sections

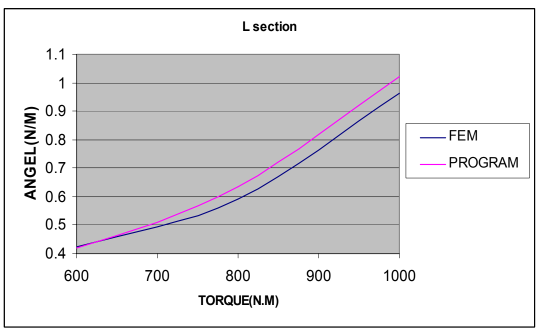

Fig 6: Torsional angle via applied moment for L sections

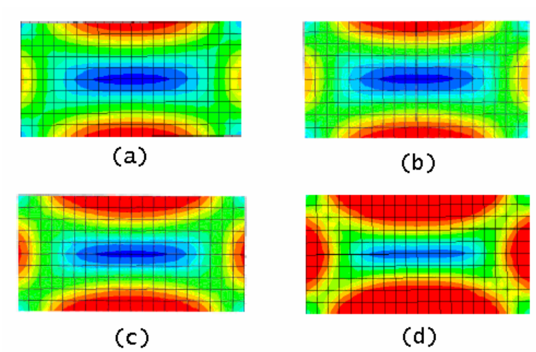

Fig 7: Progress of plastic zone in elastic area by FEM. A)1000 N.M. B)1100 N.M. C)1200 N.M. D)1300 N.M.

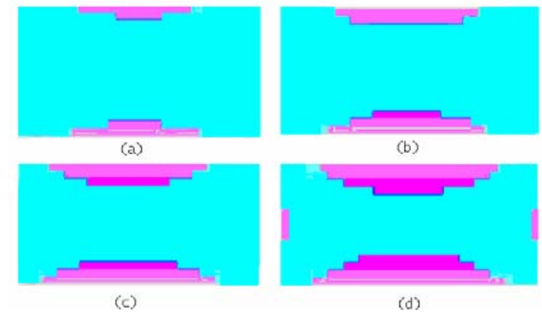

Fig 8: Progress of plastic zone in elastic area by software. A)1000 N.M. B)1100 N.M. C)1200 N.M. D)1300 N.M.

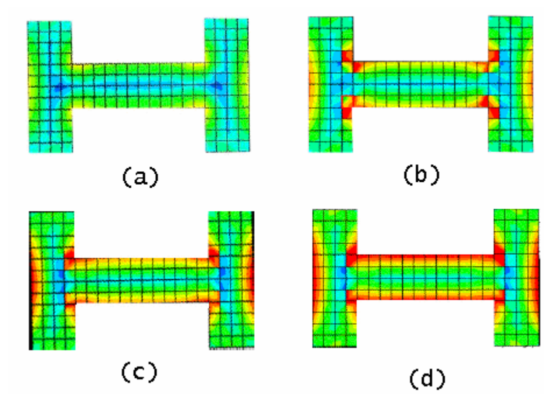

Fig 9: Progress of plastic zone in elastic area according to FEM by the following moments (N.M): a)4000 b)5000 c)5500 d)6000.

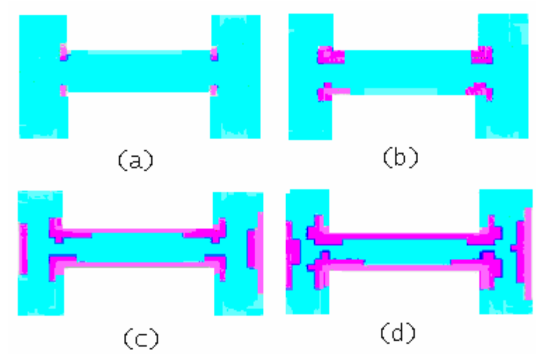

Fig 10: Progress of plastic zone in elastic area according to software by the following moments (N.M): a)4000 b)5000 c)5500 d)6000.

Fig 11: Comparison of the progress of plastic zone in elastic area by FEM and our software for (a: 600 N.M), (b: 800 N.M), (c: 1000 N.M)

It is apparent that there is a good analogy between FEM and software contours. The red region shows the plastic zones. In addition it is evident from the torsional angle via moment curve that with a desired moment the answers of the software will predict an upper bound for torsional angle. Also with decrease of moment the accuracy of solution will be raised.

4.DISCUSSION AND CONCLUSION

The consistency between results given by software and FEM approves the approximation method. In the solutions carried out previously for elastoplastic torsion of sections; all were concerning an exact analytic or numeric solution for the problem, which was ended to some solutions in oval and polygon sections; with mathematical analytic approaches. But none of which tend to make a logical assumption that relate differential equation to a simplified form. There after no outstanding progress was made after 1980. Although a constant stress assumption was made throughout of abovementioned previous tries, reconstruction of differential equation to an uncoupled construction will provide us solving for a varying yield stress. Comparing the solution done by software and FEM for different sections, the software will predict a higher torsional angle, in other words it will suggest an upper bound for a certain moment. Therefore we can use the results for design of different sections. In all of the solutions, with increasing the torsional angle the error will be reduced. In the progress of plastic zone there is a close consistency between the software and FEM results. This will lead to a precise description of progress of plastic zone to the elastic one continuously. The discontinuity pattern has unbelievable results. Stress discontinuity in non corner points is less or equal to the discontinuity computed by FEM. This will be a more powerful evidence for the certainty of the results. Finally to reach to a new, more efficient technique; simplifying the governing differential equation in order to device a simplified model for analysis, is requisite. Among mechanical engineering problems, coupled analysis is nearly the most complicated one and often there is no solution for it. In some circumstances only numerical approaches can be utilized although impractical some where else. On the other hand boundary conditions and constraints of problem on solution are not usually considered. So that it is better to grasp a new view point; taking in mind the effect of boundary conditions and the constraints on solution applied by the problem. Finally we can say that there is a variety of approaches for solving the mechanical models with coupled governing equations; FEM, mesh less finite difference methods, each has some downside such as huge computational effort, divergence of solutions and the time consumption. In the method described here the above negative effects are not present, since there is no try off process.

Accordingly with a wise elimination and revision of the boundary conditions the governing differential equations will be decoupled. Then they can be solved simultaneously, afterwards by combining the results the solution is found. In the following by utilizing appropriate approaches in treating the differential equations, there is an appealing hope to find simplifying methods for the equations and to gain some parallel equations with same results but easier to solve.

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6. NOMENCLATURE

Symbol	Meaning	Unit
G	Torsion module	(Pa)
P	Pressure	(Pa)
S_Y	Torsion module	(Pa)
θ	Pressure	(Rad)
τ	Torsion module	(Pa)
Ψ	Torsion function	(Pa/m)