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LARGE DISPLACEMENT FREE VIBRATION ANALYSIS OF ROTATING BEAM

Debabrata Das[1], **Prasanta Sahoo**[2] **and Kashi Nath Saha**[3]

Department of Mechanical Engineering, Jadavpur University, Kolkata – 700032, India.
[1]debu235@yahoo.co.in, [2]psjume@gmail.com, [3]kashinathsaha@gmail.com

ABSTRACT
The paper aims at presenting an approximate solution for large displacement free vibration problem of a linearly tapered rotating beam. The method employed requires the static solution of displacement field of the beam first and then the dynamic problem is formulated as an eigen value problem using the static solution. The method is based on energy formulation and applies minimum potential energy principle for the static problem and Hamilton's principle for the dynamic problem. The displacement field is approximated by a series of linear combination of undetermined parameters and admissible orthogonal coordinate functions. The coordinate functions are generated using Gram Schmidt scheme. The method is validated successfully with the available results and some new results are presented.

Keywords: Rotating Beam, Centrifugal Stiffening, Large Displacement.

1. INTRODUCTION
Simplified model of a rotor blade can be presented by a rotating clamped-free beam of non uniform cross section under centrifugal loading. This model can be made closer to the real system by providing an offset between the axis of rotation and initiation of the beam at the fixed end. Dynamic analysis of such mechanical component using clamped-free non uniform rotating beam model is an interesting area of research. It is well known that a non-linear analysis is necessary to predict the dynamic behavior of beams subjected to large displacement. For large displacement free vibration analysis of a beam, the non- linear natural frequencies differ significantly from their linear counterpart due to the effect of stretching of the middle plane. So, prediction of large amplitude vibration frequencies of rotating beams has got significant importance in practical applications.

Marur [1] presented an excellent review work for the development of non-linear vibration formulations of beams. Since exact solution of the governing differential equation of rotating beam vibration can not be obtained, various researchers have used different approximate methods for the analysis of such problem. Yokoyama [2] used finite element technique to study the out-of-plane free vibration behavior of rotating beams. He derived governing equations by applying Hamilton's principle and incorporated shear deformation and rotary inertia in the mathematical model. Udupa and Varadan [3] used hierarchical finite element method for the same purpose. Dynamic stiffness method and Frobenius method of series solution of differential equations had been used by Banerjee [4] to simulate the free vibration characteristics of Bernoulli-Euler beam. Using this method, he studied the dynamic behavior of rotating non-uniform beam by considering it as an assemblage of several uniform beams. He extended the same methodology for the free vibration analysis of Timoshenko beams [5]. Chakraborty et al. [6] developed a new finite element for rotating beam made of functionally graded material. The shape functions used to construct the proposed finite element are not only functions of length but also, they are functions of the beam length and element location across the beam. Wang and Wereley [7] proposed a spectral finite element method (SFEM) to develop a low-degree-of-freedom model for dynamic analysis of rotating tapered beams. The method uses semi-analytical progressive wave solutions of the governing partial differential equations and requires only one single spectral finite element to obtain any modal frequency or mode shape. Gunda and Ganguli [8] proposed a new rotating beam finite element in which the interpolating shape functions are functions of rotational speed and element position along the beam and account for the centrifugal stiffening effect. Chandiramani et al. [9] studied the free and forced vibration behavior of pre-twisted rotating composite blade using extended Galerkin Method. The effect of angular velocity and the magnitude and point of application of transverse concentrated load on the non-linear dynamic behavior of uniform rotating beam had been studied by Rout et al. [10].

Literature review reveals that dynamic analysis of rotating beam by approximate variational method is scarce. The present study employs an energy formulation where, the unknown displacement field is approximated as a finite linear combination of undetermined parameters with appropriately chosen admissible coordinate functions and the governing equations are obtained by applying variational principle.

2. MATHEMATICAL FORMULATION

The mathematical formulation is based on the assumption that the beam material is isotropic and homogeneous and follows linear elastic material behavior. The stress and strain measures are based on initial dimensions of the beam. The beam is of rectangular cross section having constant width and linearly varying thickness. Also, the beam has very small thickness when compared to its length, hence, the effect of shear deformation and rotary inertia is neglected. Fig. 1 shows two views of a tapered beam having length *L*, width *b* and root thickness h_r and free end thickness h_f. The linear thickness variation is given by,

$$h(\xi) = h_r(1-\beta\xi) \qquad (1)$$

where, $\xi = (x-R)/L$ is the normalized axial coordinate, R is the offset distance of the root of the beam from the axis of rotation and *x* is the axial coordinate and β is the parameter defining the geometry of taperness of the beam given by $(1-h_r/h_f)$. It should be noted that the effect of non-uniformity at the root of the beam has been neglected. It is to be noted further that the computation is carried out in normalized coordinate ξ.

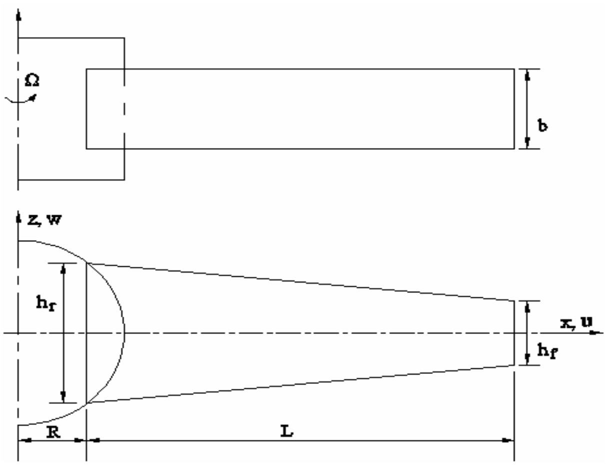


Fig 1: Projection views of a tapered beam.

The present formulation for the dynamic problem is based on displacement field which is obtained from an analysis of the beam under centrifugal loading. Using that static displacement field, the subsequent dynamic problem is formulated in terms of an eigen value problem.

2.1 Static Analysis

For static problem, the governing set of equations is obtained by the application of minimum potential energy principle which states that

$$\delta(\pi) = \delta(U+V) = 0 \qquad (2)$$

where, π is total potential energy of the system, U and V are the strain energy and work potential of the system and δ is the variational operator.

In case of large displacement analysis of beams, the axial displacement of any fibre is contributed by both bending action and stretching of midplane. The axial strain of a fibre at a distance *z* from midplane due to bending action is given by $\varepsilon_x^b = -z\dfrac{d^2w}{dx^2}$ and axial strain due to stretching of midplane is given by, $\varepsilon_x^s = \dfrac{du}{dx} + \dfrac{1}{2}\left(\dfrac{dw}{dx}\right)^2$ where, *w* and *u* denote transverse and in-plane displacements of midplane respectively. The expression for strain energy is given by,

$$U = \frac{E}{2}\int_{vol}(\varepsilon_x)^2 dv = \frac{Eb}{2}\int_R^{L+R}\int_{-\frac{1}{2}h(x)}^{\frac{1}{2}h(x)}\left(\varepsilon_x^b + \varepsilon_x^s\right)^2 dzdx\,.$$

Thus we have,

$$U = \frac{E}{2}\int_R^{L+R} I(x)\left(\frac{d^2w}{dx^2}\right)^2 dx +$$

$$\frac{E}{2}\int_R^{L+R} A(x)\left\{\left(\frac{du}{dx}\right)^2 + \frac{1}{4}\left(\frac{dw}{dx}\right)^4 + \frac{du}{dx}\left(\frac{dw}{dx}\right)^2\right\}dx \qquad (3)$$

where, E is the elastic modulus of beam material.

In arriving at Eq. (3), the following properties of beam cross section has been considered

$$b\int_{-\frac{1}{2}h(x)}^{\frac{1}{2}h(x)} dz = A(x)\,,\quad b\int_{-\frac{1}{2}h(x)}^{\frac{1}{2}h(x)} zdz = 0\,,\quad b\int_{-\frac{1}{2}h(x)}^{\frac{1}{2}h(x)} z^2dz = I(x)\,,$$

and $b\displaystyle\int_{-\frac{1}{2}h(x)}^{\frac{1}{2}h(x)} z^3dz = 0$.

where, $I(x)$ and $A(x)$ are second moment of area and cross sectional area of beam respectively. The work potential *V* of centrifugal force is given by,

$$V = \rho b\Omega^2\int_R^{L+R} h(x)\,xu\,dx \qquad (4)$$

where, Ω is the angular speed of rotation and ρ is density of beam material. The static displacements $w(\xi)$ and $u(\xi)$ are assumed as linear combinations of orthogonal functions formed by undetermined parameters as follows:

$$w(\xi) = \sum_{i=1}^{nw} d_i\phi_i(\xi)\,,\quad u(\xi) = \sum_{i=nw+1}^{nw+nu} d_i\alpha_{i-nw}(\xi) \qquad (5)$$

In (5), ϕ_i and α_i denote the set of orthogonal coordinate functions for *w* and *u* respectively and d_i represent undetermined parameters. The necessary start functions for *w* and *u* are selected to satisfy the necessary geometric boundary conditions of the beam and the higher order functions are generated from the start functions using Gram-Schmidt orthogonalization scheme. For selection of suitable start functions, results presented by Blevins [11] have been referred.

Using Eqs. (3), (4) and (5), Eq.(1) yields the governing set of equations for the static problem which is given by,

$$[K]\{d\}=\{f\} \quad (6)$$

where, $[K]=\begin{bmatrix}k_{11} & k_{12}\\ k_{21} & k_{22}\end{bmatrix}$ and $\{f\}=\begin{Bmatrix}f_{11}\\ f_{12}\end{Bmatrix}$ are stiffness matrix and load vector respectively which are of the form given below:

$$[k_{11}]=\frac{E}{L^3}\sum_{j=1}^{nw}\sum_{i=1}^{nw}\int_0^1 I\frac{d\phi_i}{d\xi}\frac{d\phi_j}{d\xi}d\xi$$

$$+\frac{E}{2L^3}\sum_{j=1}^{nw}\sum_{i=1}^{nw}\int_0^1 A\left(\sum_{i=1}^{nw}d_i\phi_i\right)^2\frac{d\phi_i}{d\xi}\frac{d\phi_j}{d\xi}d\xi$$

$$+\frac{E}{L^2}\sum_{j=1}^{nw}\sum_{i=1}^{nw}\int_0^1 A\left(\sum_{i=nw+1}^{nw+nu}d_i\alpha_{i-nw}\right)\frac{d\phi_i}{d\xi}\frac{d\phi_j}{d\xi}d\xi\ ,\ [k_{12}]=0\ ,$$

$$[k_{21}]=\frac{E}{2L^2}\sum_{j=1}^{nw+nu}\sum_{i=1}^{nw}\int_0^1 A\left(\sum_{i=1}^{nw}d_i\phi_i\right)\frac{d\phi_i}{d\xi}\frac{d\alpha_{j-nw}}{d\xi}d\xi\ ,$$

$$[k_{22}]=\frac{E}{L}\sum_{j=1}^{nw+nu}\sum_{i=1}^{nw+nu}\int_0^1 A\frac{d\alpha_{i-nw}}{d\xi}\frac{d\alpha_{j-nw}}{d\xi}d\xi\ ,\ \{f_{11}\}=0\ ,$$

$$\{f_{12}\}=\rho bL\Omega^2\sum_{j=nw+1}^{nu}\int_0^1 h(\xi)(R+L\xi)\alpha_{j-nw}d\xi\ .$$

Eq. (6) is non-linear in nature due to the presence of coupling terms and is solved by direct substitution method with relaxation parameter [12]. The effect of large transverse displacement coming from transverse loading has not been considered in this paper.

2.2 Dynamic Analysis

The governing set of equations for the dynamic problem is obtained applying Hamilton's principle which is given by,

$$\delta\left(\int_{t_1}^{t_2}(T-U-V)dt\right)=0 \quad (7)$$

where, *T, U* and *V* are kinetic energy, strain energy and work potential of external forces respectively. The expressions for *U* and *V* are given by Eqs. (3) and (4) respectively.

The expression for kinetic energy is given by,

$$T=\frac{1}{2}\rho b\int_R^{L+R} h(x)\left\{\left(\frac{\partial w}{\partial t}\right)^2+\left(\frac{\partial u}{\partial t}\right)^2\right\}dx \quad (8)$$

The dynamic displacements $w(\xi,t)$ and $u(\xi,t)$ are assumed to be separable in space and time as shown below:

$$w(\xi,t)=\sum_{i=1}^{nw}d_i\phi_i(\xi)e^{i\omega t}$$

$$u(\xi,t)=\sum_{i=nw+1}^{nw+nu}d_i\alpha_{i-nw}(\xi)e^{i\omega t} \quad (9)$$

Here, d_i represent a new set of undetermined parameters to be evaluated. The space functions are completely known from the static analysis and the set of temporal functions is expressed by $e^{i\omega t}$, where, ω represents the natural frequency of the system and $i=\sqrt{-1}$. Using these dynamic displacement fields and putting Eqs. (3), (4) (8) and (9) in Eq. (7), the governing equation of the dynamic problem can be written in the following form

$$[K]\{d\}-\omega^2[M]\{d\}=0 \quad (10)$$

where, $[M]$ is the mass matrix which is of the form given as $[M]=\begin{bmatrix}M_{11} & M_{12}\\ M_{21} & M_{22}\end{bmatrix}$ the elements of which are given by,

$$[M_{11}]=\rho b\sum_{j=1}^{nw}\sum_{i=1}^{nw}\int_0^1 h(\xi)\phi_i\phi_j d\xi\ ,[M_{12}]=0\ ,[M_{21}]=0\ ,$$

$$[M_{22}]=\rho b\sum_{j=nw+1}^{nw+nu}\sum_{i=nw+1}^{nw+nu}\int_0^1 h(\xi)\alpha_{i-nw}\alpha_{j-nw}d\xi$$

Eq. (10) can be transformed to a standard eigen value problem by suitable rearrangement which is solved numerically for calculating the natural frequencies by using IMSL routines.

3. RESULTS AND DISCUSSIONS

Table 1: Validation of non-dimensional natural frequencies for uniform cantilever beam

Mode	Present	By [8]
	λ =0	
1	3.4896	3.5160
2	22.0065	22.0345
3	61.5792	61.6972
4	120.5291	120.902
5	198.7345	199.862
Mode	λ =12	
1	13.1568	13.1702
2	37.5738	37.6032
3	79.4874	79.6146
4	140.1774	140.535
5	219.4218	220.539

Table 2: Validation of non-dimensional natural frequencies for tapered cantilever beam

λ	Present	By [8]
	Mode 1	
0	3.8216	3.8239
1	3.9755	3.9866
2	4.4329	4.4368
3	5.0782	5.0927
4	5.8707	5.8788
5	6.7336	6.7434
6	7.6479	7.6551
7	8.5853	8.5956
8	9.5479	9.5540
9	10.5121	10.5239

10	11.4905	11.5016
11	12.4701	12.4845
12	13.4617	13.4711
λ	Mode 2	
0	18.2874	18.3173
1	18.4445	18.4740
2	18.9030	18.9366
3	19.6543	19.6839
4	20.6549	20.6852
5	21.8743	21.9053
6	23.2728	23.3093
7	24.8322	24.8647
8	26.5127	26.5437
9	28.2922	28.3227
10	30.1466	30.1828
11	32.0736	32.1086
12	34.0499	34.0877
λ	Mode 3	
0	47.1639	47.2649
1	47.3171	47.4173
2	47.7727	47.8717
3	48.5188	48.6190
4	49.5457	49.6457
5	50.8360	50.9339
6	52.3672	52.4633
7	54.1145	54.2125
8	56.0633	56.1596
9	58.1869	58.2834
10	60.4993	60.5640
11	62.8880	62.9830
12	65.4279	65.5238

The results are generated using L=1.0 m, b=0.02 m, h_r=0.01 m, E=210 GPa and ρ =7850 Kg/m^3. The value of β is taken as 0.5 unless otherwise stated. Non-dimensional load λ is given by, $\Omega\sqrt{\bar{\rho}L^4/EI}$ where, $\bar{\rho}$ is the mass per unit length at the root section and non-dimensional vibration frequency $\bar{\omega}$ is given by $\omega\sqrt{\bar{\rho}L^4/EI}$. The validation of the first five natural frequencies for static condition ($\lambda = 0$) as well as with centrifugal stiffening effect ($\lambda = 12$) has been carried out with [8] for uniform cantilever beam $(\beta = 0)$ and it is tabulated in Table 1. The same for tapered cantilever beam has been shown in Table 2. Both the tables show excellent agreement, thus establishing the validity of the present method.

To visualize the effect of centrifugal stiffening on the amplitude of vibration, Fig. 2 has been presented which shows the mode shape plots for first four modes of a rotating beam with β =0.5 and R/L=0.0 both for λ =0 and λ =12. This figure clearly indicates the difference in vibration amplitude due to the effect of centrifugal stiffening. It is also indicative of the fact that effect of centrifugal stiffening is maximum on the first mode and gradually diminishes towards the higher modes.

The variation of non-dimensional out-of-plane vibration frequencies with non-dimensional speed of rotation for different R/L ratios has been shown in Figs. 3(a-e) for the first five modes respectively. Fig. 3 clearly shows the obvious fact that with increase in speed of rotation, natural frequencies increase monotonically due to the effect of centrifugal stiffening. It is clear from Fig. 3 that there is no effect of offset distance on the natural frequencies of non-rotating beam, but, for a rotating beam, the natural frequencies increase with increase in offset distance for any particular speed of rotation. In this particular study, the value of (R+L) is not kept fixed.

The effect of taper parameter β on the dynamic behavior of tapered rotating beam has been shown in Figs. 4(a-e) for first five modes respectively, in dimensionless load-frequency plane. For generating Figs. 4, the value of R/L is taken as 0.0 and the values of β are taken as 0.1, 0.2, 0.3 and 0.4. It is clear from Fig. 4(a) that there is no effect of β on the first natural frequency of rotating beam. But Figs. 4(b-e) show that with increase in the value of β, the natural frequencies of next higher modes decrease both for non-rotating as well as rotating beams. It can also be seen that the extent of decrease in vibration frequencies increases with increase in mode number.

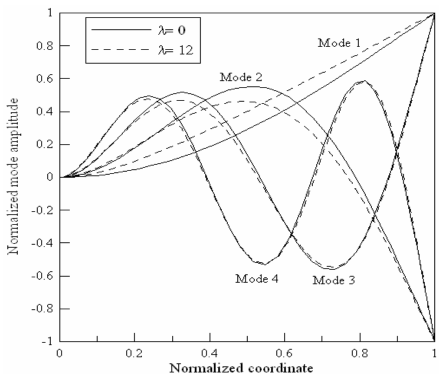
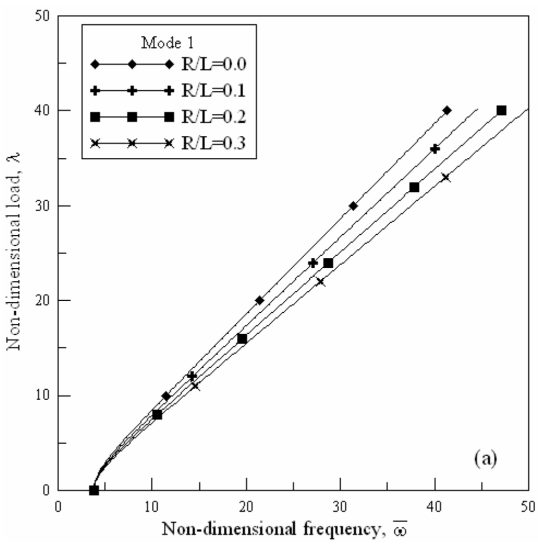
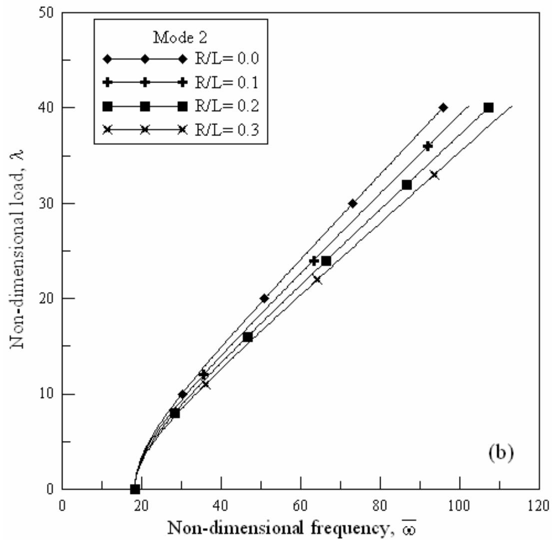


Fig 2: Effect of centrifugal stiffening on amplitude of vibration for first four modes.



(a)



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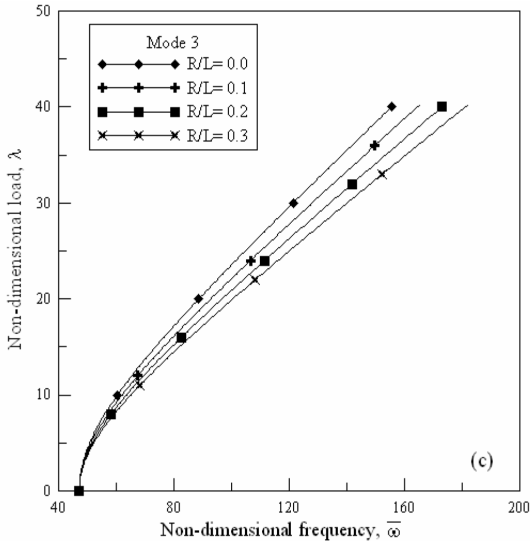

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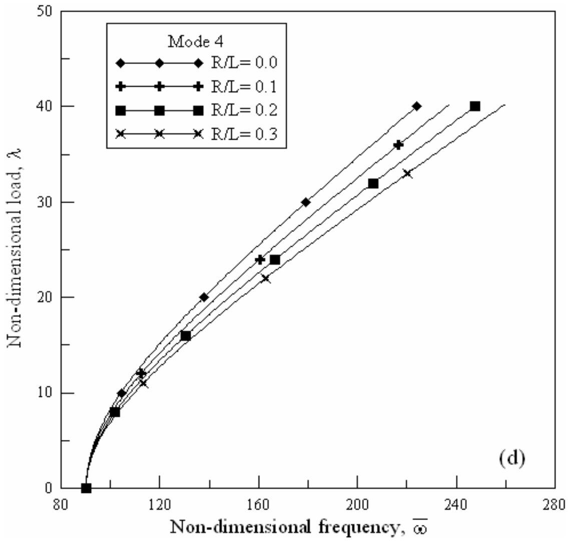

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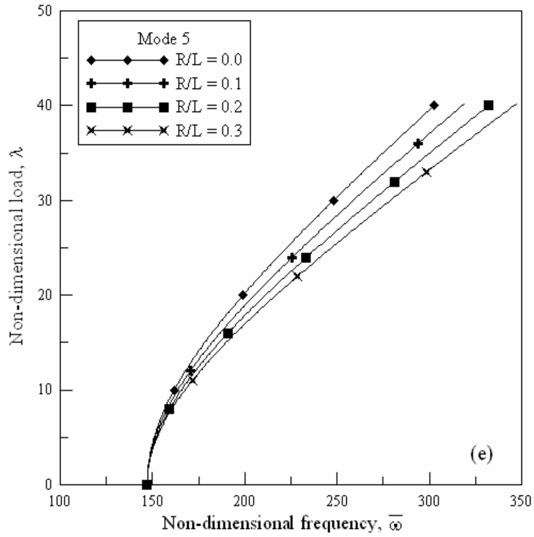
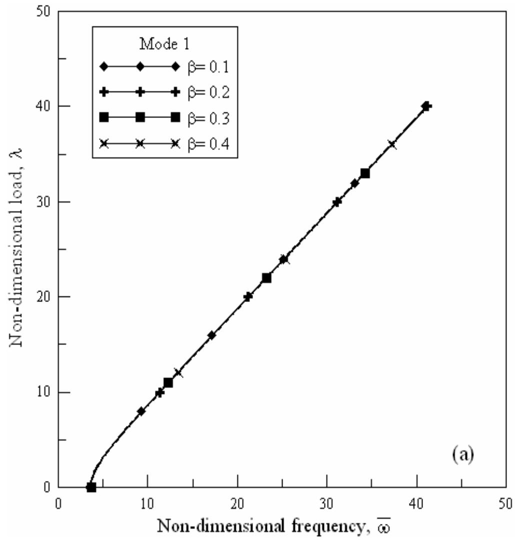
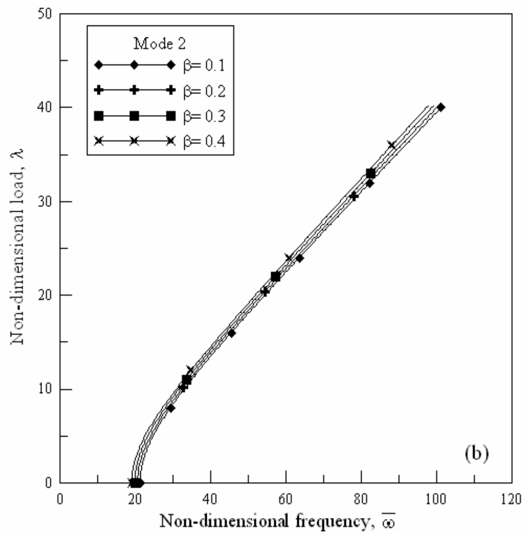


(e)

Fig 3: Effect of offset distance on the natural frequencies of tapered cantilever beam for (a) Mode 1, (b) Mode 2, (c) Mode 3, (d) Mode 4 and (e) Mode 5.



(a)



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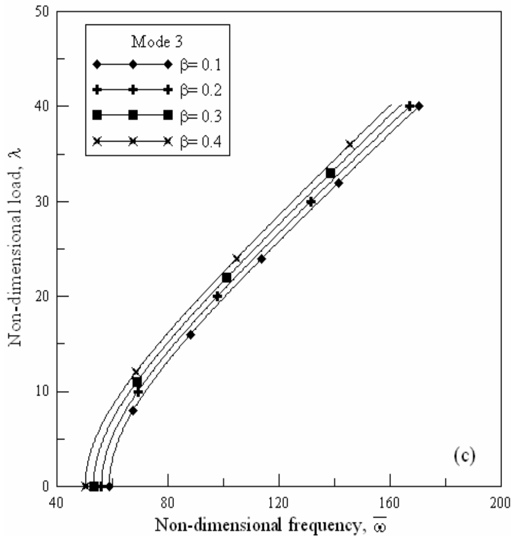

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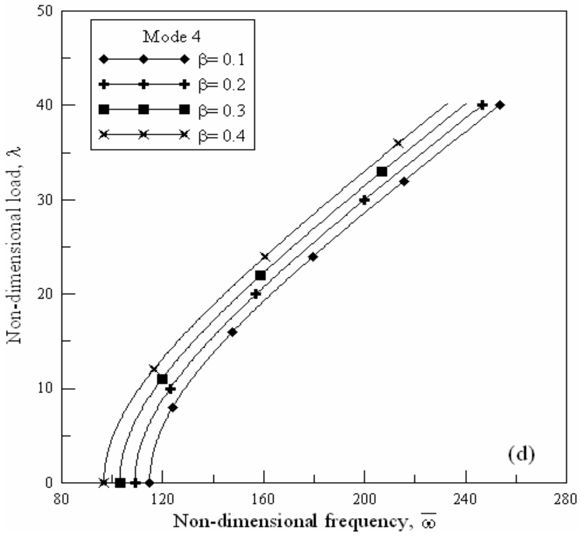
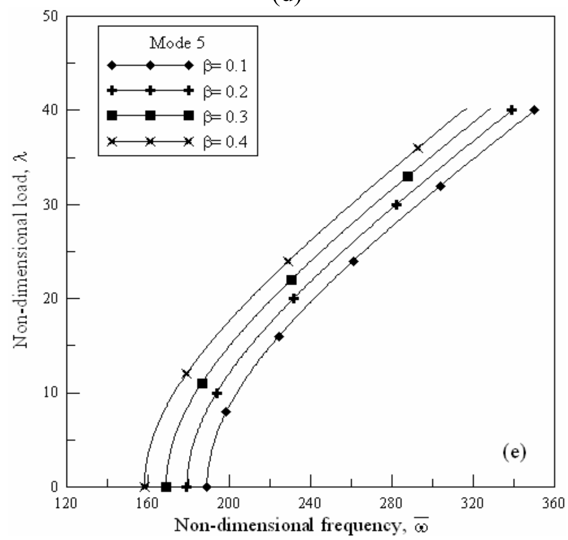


(d)



(e)

Fig 4: Effect of taper parameter β on the natural frequencies of tapered cantilever beam for (a) Mode 1, (b) Mode 2, (c) Mode 3, (d) Mode 4 and (e) Mode 5.

4. CONCLUSIONS

The present work provides a variational approach towards an approximate solution of the free vibration analysis of a tapered rotating beam. The method is validated successfully with benchmark solutions. The variation of the dynamic behavior of a rotating beam with increase in the offset distance between the axis of rotation and the root section has been shown in dimensionless load-frequency plane. The effect of taper parameter on the same has also been shown. Mode shape plots are presented to show the effect of centrifugal stiffening on the vibration amplitude. This method of analysis can be extended for other different types of non-uniform tapered rotating beams.

5. REFERENCES

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6. NOMENCLATURE

Symbol	Meaning	Unit
b	Width of beam	(m)
$\{d\}$	Unknown coefficient vector	(m)
E	Elastic modulus	(Pa)
$\{f\}$	Load vector	(N)
$h(x)$	Height of beam section	(m)
h_r	Beam height at root section	(m)
h_f	Beam height at free end	(m)
$I(x)$	Second moment of area	(m^4)
$A(x)$	Cross sectional area	(m^2)
$[K]$	Stiffness matrix	(N/m)
L	Length of beam	m
$[M]$	Mass matrix	Kg
nw	Number of functions for w	-
Nu	Number of functions for u	-
R	Offset distance	(m)
t	Time	(s)
T	Kinetic energy	(N-m)
u	In-plane displacement	(m)
U	Strain energy	(N-m)
V	Work potential	(N-m)
w	Transverse displacement	(m)
x	Axial coordinate	(m)
z	Transverse coordinate	(m)
ω	Frequency of vibration	(rad/s)
$\overline{\omega}$	Non-dimensional frequency	-
Ω	Rotational speed	(rad/s)
λ	Non-dimensional load	-
π	Total potential energy	(N-m)
β	Taper parameter	-
ϕ	Dimensionless function for w	-
α	Dimensionless function for u	-
ξ	Dimensionless coordinate	-
ρ	Density	(Kg/m^3)
$\overline{\rho}$	Mass per unit length	(Kg/m)
δ	Variational operator	-
ε_x	Axial strain (dimensionless)	-