Proceedings of the
International Conference on Mechanical Engineering 2007
(ICME2007) 29- 31 December 2007, Dhaka, Bangladesh

ICME07-AM-64

FINITE DIFFERENCE APPROACH FOR STRESS ANALYSIS IN A COMPOSITE LAMINA HAVING A CIRCULAR HOLE

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ABSTRACT
A finite difference approach is presented in this paper which can be used to investigate the two-dimensional elastic problems of a composite lamina i.e. orthotropic structure incorporated with a circular hole. Though stress function is an option for the finite difference solution, it accepts boundary conditions in terms of boundary loadings only. Here displacement potential function is used for the analysis, which enables the application of boundary conditions in terms of both stress and displacement. The stress and displacement around the hole is examined from the results of finite difference solution. In order to compare the results by the present finite difference method, another numerical technique i.e. finite element method is used. One of the existing standard commercial software (FEMLAB 3.0) is used for the finite element solution.

Keywords: Composites, circular hole, displacement potential function.

1. INTRODUCTION
Composite materials are widely used in aerospace, automobile, sports equipment, structural equipments and many other applications due to their superiorities like high strength weight ratio over monolithic materials. In many structural applications, holes of various shapes are made to meet the design requirements. The presence of holes in the components will create stress concentrations, which will reduce the mechanical strength of them. Therefore, it is of great importance to investigate state of stress around the hole in a composite lamina. For complex boundary shapes and difficulties in the management of boundary conditions of practical problems, analytical solutions become hard to achieve. Experimental methods sometimes become costly and time consuming. So the help of numerical techniques can be sought to solve these problems. Among various numerical techniques, finite difference method is being used by many researchers in their works over the years. Stress function [1] can be used for the solution of the two-dimensional elastic problems. However, problems containing boundary conditions in terms of restraints can not be discretized by the stress function formulation. Uddin, M. W. [2] proposed a formulation for the solution of two-dimensional mixed boundary value problems using the displacement potential function. Later many researchers focused their attention on the displacement potential function formulation and successfully applied for the solution of many two-dimensional elastic mixed boundary value problems [3-8]. However, these are limited to the isotropic material only. Nath, S. K. D. [9] proposed a new mathematical model to solve the problems of unidirectional orthotropic materials. Beside this, many researchers used analytical [10-11], experimental [12-13] and finite element method [14] to solve elastic problems of composite material with holes. The aim of this paper is to solve two-dimensional mixed boundary value problems of rectangular orthotropic structure with a centrally located circular hole using finite difference method based on displacement potential function.

2. MATHEMATICAL MODEL

2.1 Governing Equation
In order to provide the complete idea about the states of stresses, strains and displacements in a three dimensional body it is necessary to determine the six components of stress (σ_{xx}, σ_{yy}, σ_{zz}, σ_{xy}, σ_{yz} and σ_{zx}), six components of strain (ε_x , ε_y , ε_z , γ_{xy} , γ_{yz} and γ_{zx}). Sometimes displacement components (u_x, u_y and u_z) are evaluated instead of strain components. A unidirectional lamina falls under the orthotropic material category. If the lamina is thin and does not carry any out-of-plane loads, plane stress conditions can be assumed for the lamina. And the stress-strain relations [15] can be written by,

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{12} & S_{22} & 0 \\ 0 & 0 & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} \tag{1}$$

where, S_{ij} are called the compliance matrix constants and values of these constants are

$$S_{11} = \frac{1}{E_x}; S_{12} = -\frac{\mu_{xy}}{E_x}; S_{22} = \frac{1}{E_y}; S_{66} = \frac{1}{G_{xy}}$$

Again for the static equilibrium of the lamina, there are two equations of equilibrium (Eq. 2 and 3) and a compatible equation (Eq. 4) which must be satisfied at all points throughout the volume of the body.

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = 0 \tag{2}$$

$$\frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{xy}}{\partial x} = 0 \tag{3}$$

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} \tag{4}$$

By using the relations from Eq. 1 this compatibility equation (Eq. 4) can be expressed in terms of stress components as,

$$\left(\frac{1}{E_x}\frac{\partial^2 \sigma_{xx}}{\partial y^2} + \frac{1}{E_y}\frac{\partial^2 \sigma_{yy}}{\partial x^2}\right) - \frac{\mu_{xy}}{E_x}\left(\frac{\partial^2 \sigma_{yy}}{\partial y^2} + \frac{\partial^2 \sigma_{xx}}{\partial x^2}\right) = \frac{1}{G_{xy}}\frac{\partial^2 \sigma_{xy}}{\partial x \partial y} \tag{5}$$

Substituting the stress components in the Eqs. 2, 3 and 5 by the displacement components (u_x and u_y) it is found that Eq. 5 is surplus and the Eqs. 2 and 3 become

$$\left(\frac{E_x^2}{E_x - \mu_{xy}^2 E_y}\right)\frac{\partial^2 u_x}{\partial x^2} + \left(\frac{\mu_{xy} E_x E_y}{E_x - \mu_{xy}^2 E_y} + G_{xy}\right)\frac{\partial^2 u_y}{\partial x \partial y} + G_{xy}\frac{\partial^2 u_x}{\partial y^2} = 0 \tag{6}$$

$$\left(\frac{E_x E_y}{E_x - \mu_{xy}^2 E_y}\right)\frac{\partial^2 u_y}{\partial y^2} + \left(\frac{\mu_{xy} E_x E_y}{E_x - \mu_{xy}^2 E_y} + G_{xy}\right)\frac{\partial^2 u_x}{\partial x \partial y} + G_{xy}\frac{\partial^2 u_y}{\partial x^2} = 0 \tag{7}$$

Although these two equations are sufficient for the solution, it is still difficult to solve for two functions simultaneously. This difficulty can be overcome if these two equations can be transformed into a single equation of a single function. So to reduce the number of governing differential equations, in a single equation, a new function called displacement potential function (ψ) is defined as a function of displacement components as[9],

$$u_x = \frac{\partial^2 \psi}{\partial x \partial y} \tag{8}$$

$$u_y = -\frac{1}{Z_{11}}\left[E_x^2 \frac{\partial^2 \psi}{\partial x^2} + G_{xy}\left(E_x - \mu_{xy}^2 E_y\right)\frac{\partial^2 \psi}{\partial y^2}\right] \tag{9}$$

where, $Z_{11} = \mu_{xy} E_x E_y + G_{xy}\left(E_x - \mu_{xy}^2 E_y\right)$. With these definitions of $\psi(x, y)$, the first of the two equilibrium equations (Eq. 6) is automatically satisfied. Therefore, $\psi(x, y)$ has to satisfy the second equation (Eq. 7). This second equation in terms of $\psi(x, y)$ can be expressed as [9],

$$\left[\begin{array}{l} E_x G_{xy}\dfrac{\partial^4 \psi}{\partial x^4} + E_y(E_x - 2\mu_{xy} G_{xy})\dfrac{\partial^4 \psi}{\partial x^2 \partial y^2} \\ + E_y G_{xy}\dfrac{\partial^4 \psi}{\partial y^4} \end{array}\right] = 0 \tag{10}$$

Therefore, the problem is reduced to the evaluation of a single variable $\psi(x, y)$ from the bi-harmonic partial differential equation (Eq. 10) and this equation is the only governing equation, in this approach, for the solution of two-dimensional unidirectional composite lamina.

2.2 Boundary Conditions

To analyze the state of stress in two-dimensional arbitrary shaped body, the normal and tangential components of displacement and stress should be evaluated in terms of $\psi(x, y)$. The normal and tangential components of displacement can be found easily by simple manipulation of the Eq. 8 and 9 as follows,

$$\begin{aligned} u_n &= u_x l + u_y m \\ &= l.\frac{\partial^2 \psi}{\partial x \partial y} - \frac{m}{Z_{11}}\left[E_x^2 \frac{\partial^2 \psi}{\partial x^2} + G_{xy}\left(E_x - \mu_{xy}^2 E_y\right)\frac{\partial^2 \psi}{\partial y^2}\right] \end{aligned} \tag{11}$$

$$\begin{aligned} u_t &= u_y l - u_x m \\ &= -l.\frac{1}{Z_{11}}\left[E_x^2 \frac{\partial^2 \psi}{\partial x^2} + G_{xy}\left(E_x - \mu_{xy}^2 E_y\right)\frac{\partial^2 \psi}{\partial y^2}\right] - m\frac{\partial^2 \psi}{\partial x \partial y} \end{aligned} \tag{12}$$

Similarly, the normal and tangential components of stress can be written as the following form

$$\begin{aligned} \sigma_n &= \sigma_{xx} l^2 + \sigma_{yy} m^2 + 2\sigma_{xy} lm \\ &= l^2 \frac{E_x G_{xy}}{Z_{11}}\left[E_x \frac{\partial^3 \psi}{\partial x^2 \partial y} - \mu_{xy} E_y \frac{\partial^3 \psi}{\partial y^3}\right] \\ &\quad + m^2 \frac{E_x E_y}{Z_{11}}\left[\left(\mu_{xy} G_{xy} - E_x\right)\frac{\partial^3 \psi}{\partial x^2 \partial y} - G_{xy}\frac{\partial^3 \psi}{\partial y^3}\right] \\ &\quad - 2lm\frac{E_x G_{xy}}{Z_{11}}\left[E_x \frac{\partial^3 \psi}{\partial x^3} - \mu_{xy} E_y \frac{\partial^3 \psi}{\partial x \partial y^2}\right] \end{aligned} \tag{13}$$

$$\begin{aligned} \sigma_t &= \sigma_{xy}(l^2 - m^2) + (\sigma_{yy} - \sigma_{xx}) lm \\ &= -(l^2 - m^2)\frac{E_x G_{xy}}{Z_{11}}\left[E_x \frac{\partial^3 \psi}{\partial x^3} - \mu_{xy} E_y \frac{\partial^3 \psi}{\partial x \partial y^2}\right] \\ &\quad + lm\frac{E_x E_y}{Z_{11}}\left[\left(\mu_{xy} G_{xy} - E_x\right)\frac{\partial^3 \psi}{\partial x^2 \partial y} - G_{xy}\frac{\partial^3 \psi}{\partial y^3}\right] \\ &\quad - lm\frac{E_x G_{xy}}{Z_{11}}\left[E_x \frac{\partial^3 \psi}{\partial x^2 \partial y} - \mu_{xy} E_y \frac{\partial^3 \psi}{\partial y^3}\right] \end{aligned} \tag{14}$$

Now all possible boundary conditions (u_n, u_t, σ_n and σ_t) are evaluated in terms of $\psi(x, y)$. So in the region of study the governing differential equation (Eq. 10) should be applied at all node points except the boundary node points and boundary conditions (Eq. 11 to 14) should be applied at boundary node points.

2.3 Description of the Configuration

A rectangular composite lamina with a centrally located circular hole under uniform tensile loading is shown in the Fig. 1 where b/a=2, r/a=0.50 and x-direction is considered as the fiber direction. Considering the symmetry of the configuration, one quarter (top-right) of the lamina is chosen for the study, as shown in Fig. 2. The material considered for the lamina is boron reinforced epoxy matrix (Boron/ Epoxy) composite. The properties of the material are taken as, E_x=204 GPa, E_y=18.5 GPa, G_{xy}=5.59 GPa, μ_{xy}=0.23. And the boundary conditions required for the problem are shown in the Fig. 3, where $\sigma_x^o = 2.4\times10^6$ N/m^2.

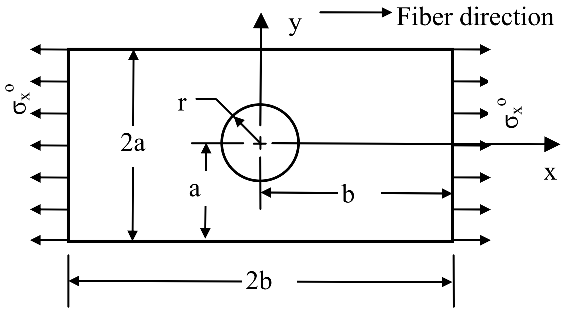

Fig 1: Geometry of the actual problem

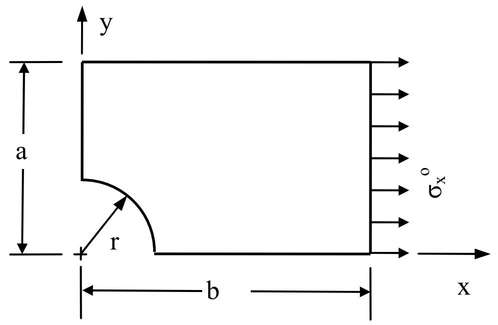

Fig 2: One quarter model of the composite lamina

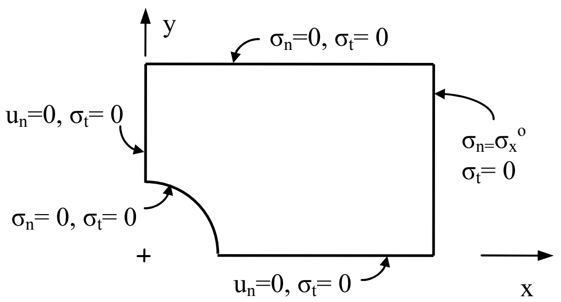

Fig 3: Boundary conditions applied in the problem

3. SOLUTION PROCEDURE

For the finite difference solution, the whole region is divided into meshes with lines parallel to the rectangular co-ordinate axes, which results in a finite number of node points. Finite Difference form of the governing equation is applied to the inner node points and boundary conditions are then applied to the boundary node points. But difficulties arises incase of boundary node points. Firstly for curved or irregular boundary, boundary points may not coincide with the field grid points and secondly each boundary node point has to satisfy at least two boundary conditions. To overcome second problem a boundary near the physical boundary is assumed to exist which is named as imaginary boundary. Each physical node point has to be provided with an imaginary node point and it will be the immediate outward field grid point, e.g. for a node point at the outer top boundary the imaginary boundary point will be the immediate top field grid point. To solve the first problem a new boundary is incorporated which is called reference boundary [10]. If any physical hole boundary point matches with the rectangular field grid point, then the point itself will act as the reference boundary point. When the physical hole boundary point does not match with the rectangular grid point then the nearest grid point will be the reference boundary point of that physical boundary point. By this the actual or physical boundary is transformed into the reference boundary which gives a new region and will be the region under consideration. For the curved surface in the hole region the physical, reference and imaginary boundary node points are shown in the Fig. 4.

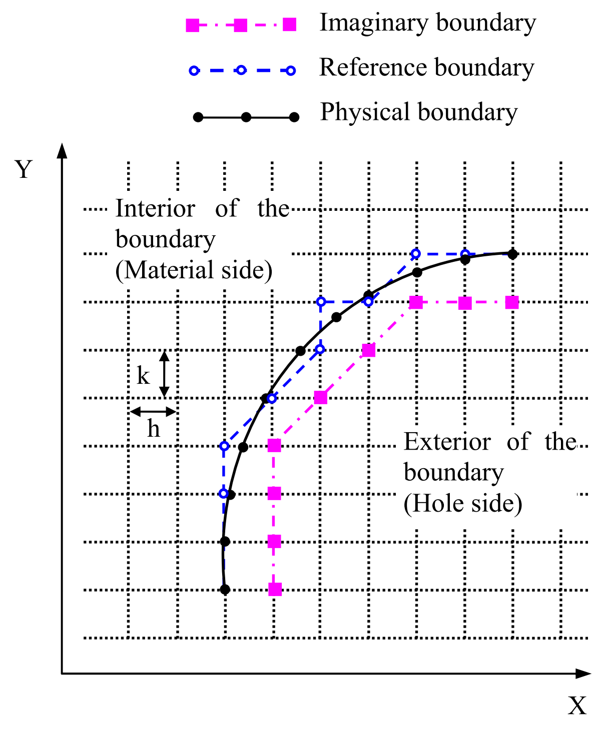

Fig 4: Imaginary, reference and physical boundary of hole boundary points

So when finite difference equations are applied to the node points of the whole region, it gives a set of linear algebraic equations equal to the number of total node points in the region. In these equations only unknowns are the ψ's. Here L-U decomposition method is used to solve the set of equations and hence value of ψ at each node point is found. Once values of ψ's are evaluated the stress and displacement components at each point can be found from the finite difference form of the equations (Eq. 11 to 14) presented before.

4. RESULTS AND DISCUSSION

4.1 Distribution of Displacement

Distribution of the normalized longitudinal displacement (u_x/a) with respect to y-axis (y/a) at different vertical sections of the body is illustrated in Fig. 5. At the left edge (x/b=0.0), midsection of the hole, the normalized longitudinal displacement is found to be zero, which confirms the applied boundary condition. And it is observed that displacement increases gradually in the positive x direction for a particular horizontal section and all points of the body tends to move towards right due to the applied load. For any vertical section, the trend of the distribution shows the presence of a hole in the lamina. Maximum displacement is found at the bottom (y/a=0.0) of the extreme right section (x/b=1.0). Fig. 6 shows the distribution of the normalized transverse displacement (u_y/a) with respect to y-axis (y/a) at different vertical sections of the lamina. For the vertical sections from x/b=0.0 to 0.5 the magnitude of the transverse displacement is negative and for the other two sections (x/b=.75 and 1.0) it is positive. Positive value means displacement along positive y-direction and negative value means displacement along negative y direction. Therefore signs of the displacement suggest that from left to almost middle section, all points tend to move in downward direction and after mid-section all points tend to move in upward direction. It is also observed that at the bottom edge (y/a=0.0) transverse displacement equals zero, which confirms the applied boundary condition.

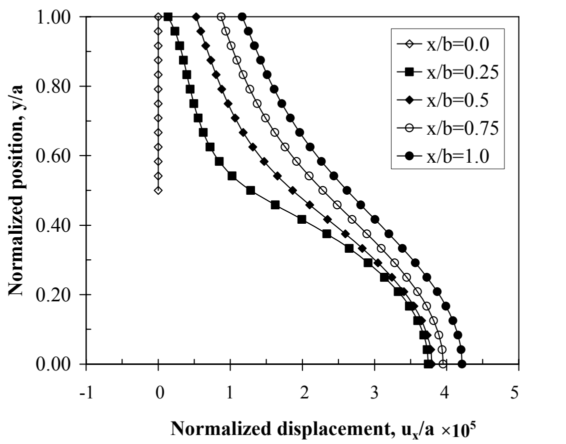

Fig 5: Distribution of normalized displacement u_x/a

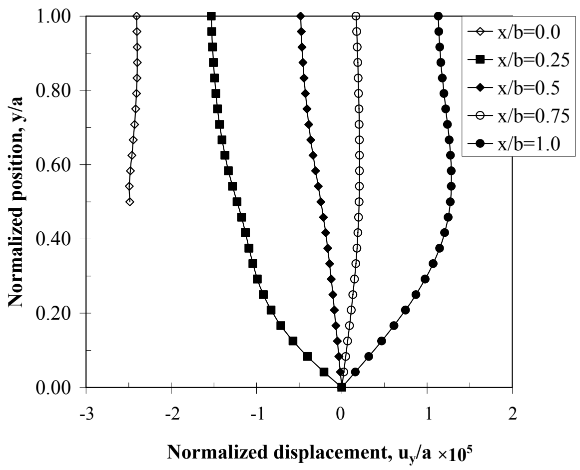

Fig 6: Distribution of normalized displacement u_y/a

4.2 Distribution of Stress

In a view to present the graphs of stress in dimensionless form, actual values of stress are divided by the applied stress. Fig. 7 illustrates the normalized axial normal stress ($\sigma_x/\ \sigma_x^o$) with respect to y-axis (y/a). All the points of the body have positive magnitude for this stress component and positive sign of the stress means stress along the outward direction of the body. At the extreme right end, the magnitude of normalized axial stress equals to unity, i.e. actual axial is equal to the applied axial stress. In all sections except x/b=0.0 for the points beyond y/a= 0.42 the axial normal stress exceeds the applied load. However, from the viewpoint of this axial stress, most critical section is left edge (x/b=0.0). At y/a=0.5 of this section (top of the hole) the magnitude of axial stress is more than ten times of the applied stress. Fig. 8 illustrates the normalized lateral stress ($\sigma_y/\ \sigma_x^o$) distribution with respect to y-axis at different sections of the lamina. Maximum lateral stress is occurred at the bottom of the right edge and side of the hole. In other sections of the body lateral stress is very insignificant as compared to the axial stress. Fig. 9 shows the distribution of normalized shear stress ($\sigma_{xy}/\ \sigma_x^o$) with respect to y-axis. Parabolic nature of the curves is observed for the mid-region (x/b=0.5 and 0.75) of the body and a very distinct trend is found for the section x/b=0.25. Important factor to observe that magnitude of lateral and shear stress at any point of the body does not exceed the magnitude of the applied stress.

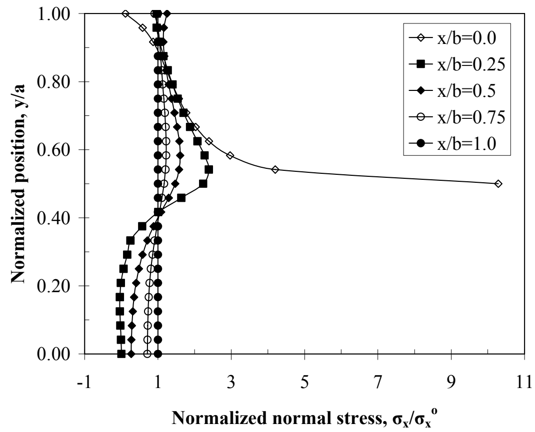

Fig 7: Distribution of normalized axial stress

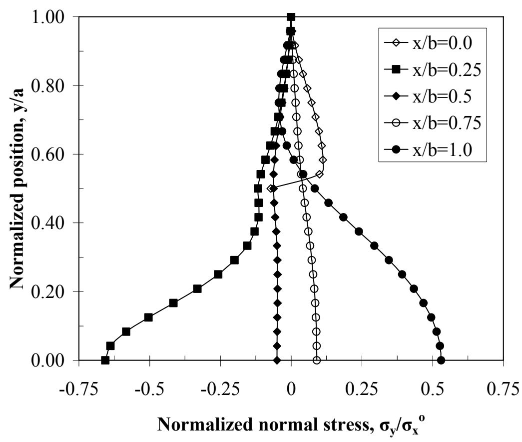

Fig 8: Distribution of normalized lateral stress

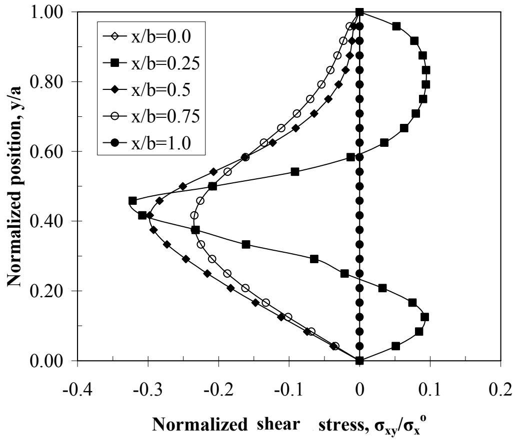

Fig 9: Distribution of normalized shear stress

4.3 Comparison with Finite Element Solution

Commercial software FEMLAB 3.0 is used for the solution of the problem. Two critical sections are selected for the study, which are x/b=0.0 (left edge) and x/b=1.0 (right edge). Fig. 10 to 14 shows the comparison of the results found from the finite difference and finite element method and represents the distribution of normalized longitudinal displacement (u_x/a), normalized transverse displacement (u_y/a), normalized axial normal stress (σ_x/σ_x^o), normalized lateral normal stress (σ_y/σ_x^o) and normalized shear stress (σ_{xy}/σ_x^o) with respect to y-axis respectively for those two sections. It is observed that the results agree well within the acceptable limit.

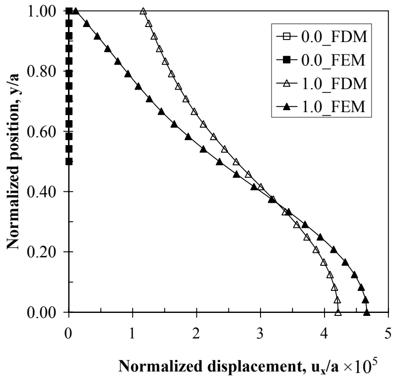

Fig 10: Comparison of normalized displacement u_x/a

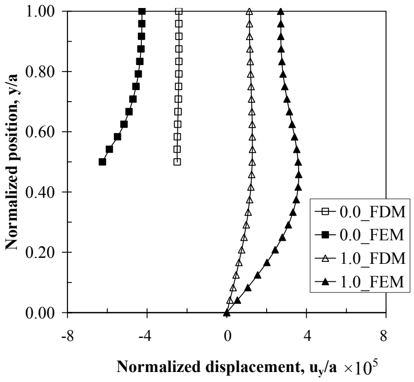

Fig 11: Comparison of normalized displacement u_y/a

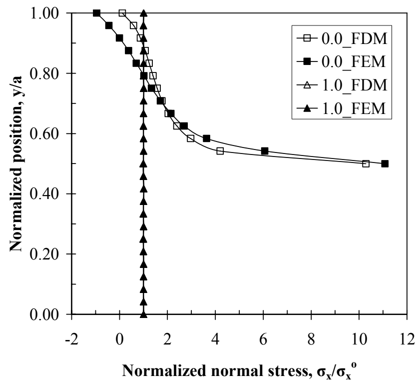

Fig 12: Comparison of normalized axial stress

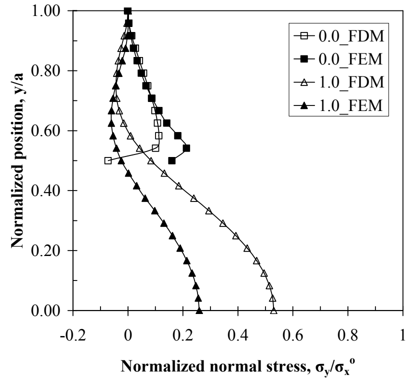

Fig 13: Comparison of normalized lateral stress

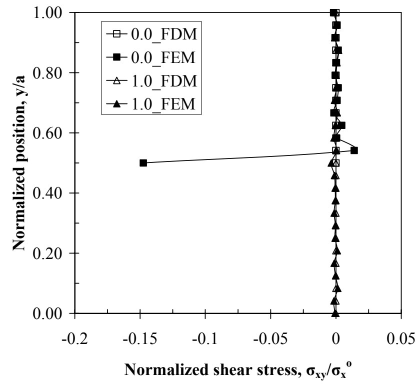

Fig 14: Comparison of normalized shear stress

5. CONCLUSIONS

The finite difference approach depicted in this paper has been developed for the solution of the two-directional composite lamina with a hole. However, it can be used for any composite material, loading conditions and geometrical shape. The results of finite difference and finite element methods are found to be in good accord, which extends as well as confirms the capability the displacement potential function formulation.

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7. NOMENCLATURE

Symbol	Meaning
E_x	Elastic modulus of the material in x-direction (Fiber direction)
E_y	Elastic modulus of the material in y-direction
μ_{xy}	Major Poisson's ratio
G_{xy}	In-plane shear modulus in the x-y plane
Ψ	Displacement potential function
u_x, u_y	Displacement components in the x- and y-direction
ε_x, ε_y	Strain components in the x- and y-direction
σ_{xx}, σ_{yy}	Normal stress components in the x- and y-direction
σ_{xy}	Shearing stress component in the *xy* plane
u_n, u_t	Displacement components in the normal and tangential direction
σ_n, σ_t	Stress components in the normal and tangential direction
σ_x^o	Applied stress in x-direction
a, b	Dimensions of the rectangular lamina in y- and x-directions, respectively