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EFFECTS OF THERMAL STRESSES AND BOUNDARY CONDITIONS ON THE RESPONSE OF A RECTANGULAR ELASTIC BODY MADE OF FGM

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ABSTRACT

Response of an FGM elastic body, under variation of temperature (inducing thermal stresses) and two different boundary conditions, is predicted using a computer code in *C*. For this purpose, a rectangular FGM elastic body is chosen and the stresses induced in the body are analyzed by solving the boundary value problem using the finite element method (FEM), a numerical technique widely used by engineers to solve problems of solid and structural mechanics. It is assumed that the Young's modulus varies either linearly or exponentially in the vertical direction. The coefficient of thermal expansion and Poisson's ratio are considered to be constant. The analysis is carried out by using both the triangular element (also known as the constant strain triangle, CST) and the rectangular element. The results obtained for the two types of elements are then compared.

Keywords: FGM, Boundary Value Problem, Symmetric and Asymmetric Boundary Conditions, Thermal Stress, FEM

1. INTRODUCTION

Functionally graded materials (FGMs) have greatly attracted the attention of the researchers in recent times. These materials have certain advantages over the homogeneous materials and conventional composite materials. To make the best use of the FGMs, a thorough understanding of their behavior in different conditions is necessary.

Functionally graded materials (FGMs) are non-homogeneous solids, which consist of two or more distinct material phases, such as different ceramics or ceramics and metals, and are the mixture of them such that the composition of each changes continuously with space variables. The change in composition induces material and micro structural gradients, and makes the functionally graded materials different in behavior from homogeneous materials and conventional composite materials [1-4].These materials are tailorable in their properties via the design of the gradients, which, in turn, depend on material distributions. From a mechanics viewpoint the main advantages of material property grading appear to be improved bonding strength, toughness, wear and corrosion resistance and reduced residual and thermal stresses. Some typical applications include thermal barrier coatings of high temperature components in gas turbines, surface hardening for tribological protection and graded interlayers used in multilayered microelectronic and optoelectronic components [1-4]. Because of their outstanding advantages over homogeneous materials and conventional composite materials, in recent years, much attention has been paid for analyzing the various aspects of FGMs to get an in depth knowledge of the potential applications of these materials as structural and functional elements in aerospace industries, chemical industries, and nuclear power plants. Therefore, FGMs have become very important in the field of materials science research.

For the ease of analysis, it is often conventional to regard the material properties to be some certain assumed functions of the space variables. The most common functions assumed to model the material properties are the exponential function, the power function and the linear function.

In this study, it is assumed that the rectangular elastic body is made of a functionally graded material (FGM) and the Young's modulus of the material varies either linearly or exponentially in the vertical direction. Both the coefficient of thermal expansion and the Poisson's ratio remain constant. The body is subjected to boundary conditions (both symmetric and asymmetric) and variation of temperature and the stresses induced due to these are analyzed by the finite element method.

The finite element method (FEM) is a numerical method which can be used for the accurate solution of complex engineering problems. The finite element method as known today has been presented in 1956 by Turner et al. [5]. The general applicability of the FEM can be seen by observing the strong similarities that exist between various types of engineering problems [5,6]. Considering all these facts, the finite element method is chosen here to analyze the stresses.

All the elasticity problems can be dealt within the framework of any one of the following three fundamental boundary value problems:

1. Determination of the elastic field (stress and displacement) in the interior of an elastic body in equilibrium when the boundary conditions are prescribed in terms of forces and stresses only.
2. Determination of the elastic field in the interior of an elastic body in equilibrium when the boundary conditions are prescribed in terms of strains or displacements only.
3. Determination of the elastic field in the interior of an elastic body in equilibrium when the boundary conditions are prescribed in terms of stresses over a part of the boundary and displacements over the other parts of the boundary. These categories of problems are called mixed boundary value problems [7].

In our study, we considered a displacement boundary value problem.

As a representative elastic body, a 2D rectangular plate made of FGM is selected (Fig. 1). This is because 2D rectangular bodies are the most commonly used structures. Different boundary conditions can be applied along the four edges of the body.

2. MATHEMATICAL MODELING

Consider a rectangular body made of FGM and having length L, width W and thickness t respectively. The body is regarded as a 2-dimensional (2D) elastic body because its thickness is very small compared to its length and width. The body is represented in the Cartesian coordinate system (Fig 1).

The Young's modulus, Poisson's ratio and coefficient of thermal expansion are denoted by E, ν and α respectively. It is assumed that the Poisson's ratio ν and coefficient of thermal expansion α remain constant. Only the Young's modulus E varies either linearly or exponentially in the vertical (y)direction. Therefore, in equation form,

$E = E_0 - \beta * y$ (linear case) …… (1)

$E = E_0 * \exp(-\beta * y)$ (exponential case) …. (2)

Where E_0 and β are constants. The variation of temperature is denoted by ΔT and horizontal and vertical displacements are denoted by u and v respectively. Because of the effect of ΔT and imposed values of u and v (that is, displacement boundary conditions) the body will undergo strains and as a result stresses will develop.

For the above model of the problem, the developed stresses are estimated by the finite element method. Since analytic solution of FGM problems are often difficult to obtain, the FE analysis is done by using both the triangular element and the rectangular element so that some comparisons can be made.

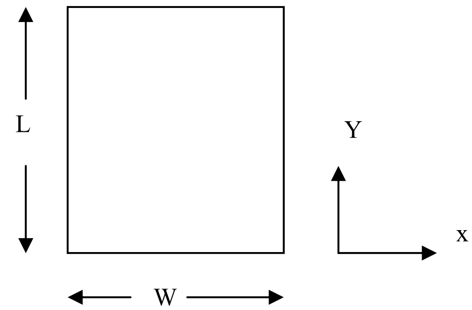
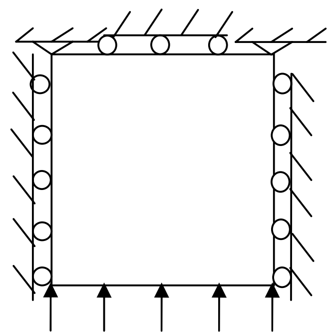


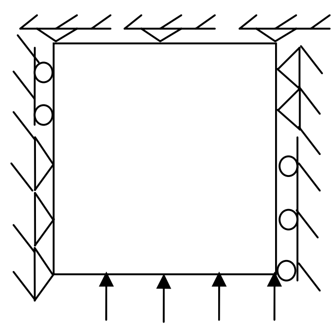
Fig 1: A 2D rectangular elastic body

2.1 Description Of The Particular Problem



v = 0.1mm

Fig 2: Symmetric boundary condition



v = 1mm

Fig 3: Asymmetric boundary condition

In order to get some numerical results, some specific values are considered for the dimensions of the body, the properties of the materials and the imposed conditions. Here we consider,

Length of the body, L = 1000 mm
Width of the body, W = 800 mm
Thickness of the body, t = 2 mm
Young's modulus, E_0 = 200 GPa
β = 80000 MPa / m (for the linear case), β = 1.0 MPa / m (for the exponential case).

Poisson's ratio, $\nu = 0.3$
Coefficient of thermal expansion, $\alpha = 0.000012\ /^0C$
Increase of temperature, $\Delta T = 20^0C$
So, from equations (1) and (2), E = 200000 - (80000 * y) MPa (for the linear case)
E = 200000 * exp (- y) (for the exponential case)
where y is in meters in the vertical direction.

The two types of boundary conditions used, namely symmetric and asymmetric are shown in Figs. 2,3. Applied displacements (v=0.1mm for symmetric boundary conditions and 1mm for asymmetric boundary conditions) are the applied loads in addition to temperature variations for four different cases.

2.2. Solution by FEM

2.2.1 Stress – strain relations

Since the thickness of the body is almost negligible compared with the other two dimensions, the analysis of the thin body loaded in the plane of the body can be made using the assumption of plane stress. In plane stress distribution, it is assumed that

$$\sigma_{zz} = \sigma_{zx} = \sigma_{yz} = 0 \qquad (3)$$

The strain & stress vector are expressed as

$$\boldsymbol{\varepsilon} = \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \end{Bmatrix} , \quad \boldsymbol{\sigma} = \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} \qquad (4)$$

When ΔT is nonzero, there will be initial (thermal) strain ε_0.The thermal strain vector is expressed as

$$\boldsymbol{\varepsilon_0} = \alpha * \Delta T * \begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix} \qquad (5)$$

The material property matrix [D] is expressed as

$$[\,D\,] = \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}$$

The stress is calculated as $\boldsymbol{\sigma} = [\,D\,] * \boldsymbol{\varepsilon}$ (6)

Strains are defined as $\varepsilon_{xx} = \frac{\partial u}{\partial x}$, $\varepsilon_{yy} = \frac{\partial v}{\partial y}$,

$$\varepsilon_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \qquad (7)$$

where u and v are nodal displacements along horizontal and vertical direction respectively.

The strain vector is expressed as $\boldsymbol{\varepsilon} = [\,B\,]\ \mathbf{u}$ (8)

[B] is a 3*6 matrix for triangular element and 3*8 matrix for rectangular element.

2.2.2. Step by step solution

The initial step was to divide the structure into elements and to choose the suitable interpolation function. Considering the 2D rectangular body, the triangular element (also known as the membrane element) was selected. Another type of element was required to make comparison and to provide a more reliable result, if possible. Because of the geometry of the body, the rectangular element was a natural choice. The triangular element is a 2D simplex element as the interpolation function is a linear function. On the other hand, the rectangular element is a multiplex element because its boundaries are parallel to the coordinate axes to achieve inter element continuity. Here 40 triangular elements and

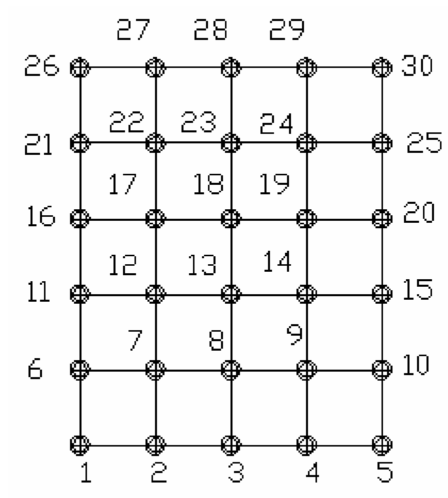


Fig 4: Node numbering.

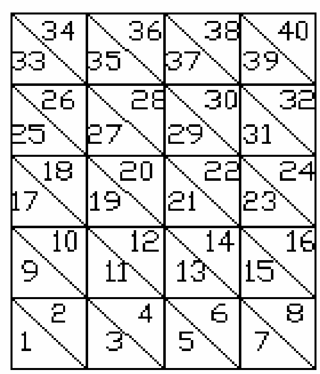


Fig. 5: Element numbering for triangular element

20 rectangular elements were used in 5 layers for the 2D FGM body (Figs. 4,5). The triangular element was taken as half of rectangular element. The elements were of equal size. The number of nodes was 30.The FEM provides better approximation to the exact value, if the aspect ratio (the ratio of the largest dimension of the element to the smallest dimension) is nearly unity [6].Here, the aspect ratio was unity for the rectangular element and 1.41 for the triangular element, respectively (Figs. 4,5).The bottom layer is the layer 1 and the top layer is the layer 5. For clarity, layer 1 and layer 2 contain nodes 6-10 and 11-15, respectively, and so on, in Fig. 4. The next step was to derive the element equations that is,

to formulate the element stiffness matrix and load vectors. These equations can be obtained using the principle of minimum potential energy. The following formulae were used,

Element stiffness matrix,

$$[k^{(e)}] = \iiint_{V^{(e)}} [B]^T [D][B].dV \quad (9)$$

Element load vector due to thermal strain,

$$\mathbf{P}_i^{(e)} = \iiint_{V^{(e)}} [B]^T [D]\, \varepsilon_0 .dV \quad (10)$$

Then the element equations were assembled to form the overall system of equations of the form

$$[\ k\ ]\ \mathbf{u}\ =\ \mathbf{P} \quad (11)$$

where [k] is the assembled stiffness matrix, **u** is the vector of nodal displacements and **P** is the vector of nodal forces for the complete structure.

The next step was the incorporation of boundary conditions. Here, we had only displacement boundary conditions of symmetric and asymmetric types (figs. 2,3).

Then the system of equations was solved by Gaussian elimination to obtain the nodal displacements **u**. From these known nodal displacements, element strains and stresses were computed using the relations of plane stress conditions.

3. RESULTS AND DISCUSSION

Regarding validity of the FEM modeling of the problem, a sample test was done for an isotropic, homogenous elastic body of the same size under constant stress condition. An exact result was obtained by the same code used later for this specific problem.

Case 1. No thermal stress & symmetric boundary condition

The triangular and rectangular element gave absolutely identical results for the symmetric case. The strains are different for the 5 layers but the stresses are same for all the elements. This is true for both linear and exponential variation of Ethel results are shown in the table below:

Table 1: Stresses for the FGM for case 1

Variation of E	σ_{xx} (MPa)	σ_{yy} (MPa)	σ_{xy} (MPa)
Linear	-5.17	-17.22	0
Exponential	-3.84	-12.81	0

Case 2. No thermal stress & asymmetric boundary conditions

The results provided by the triangular and rectangular element are different. Both the strains and the stresses vary from element to element for linear as well as exponential variation of E. Specific results are not shown for brevity.

Case 3. With thermal stress & symmetric boundary conditions

For the triangular element, the stresses and strains are different in different layers. But at a particular layer, the values of strains and stresses are constant. The exponential case shows more variation than the linear case.

For the rectangular element, the stresses and strains are different in each element for both linear and exponential case.

One important thing to be noted here is that, the analysis with triangular elements shows that no shear stresses will develop in the body. But in case of rectangular elements, shear stresses do exist and vary from element to element.

The results of triangular and rectangular elements differ largely. Some results are shown graphically (Figs. 6,7).

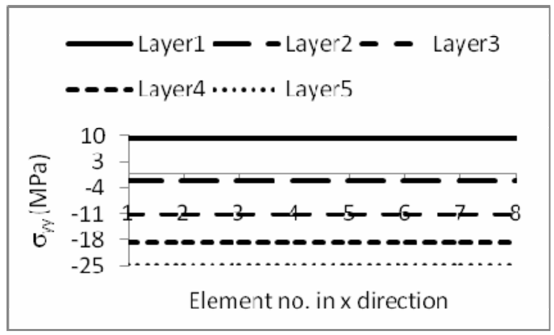


Fig 6:Stress distribution for case 3 with triangular element (E varies exponentially).

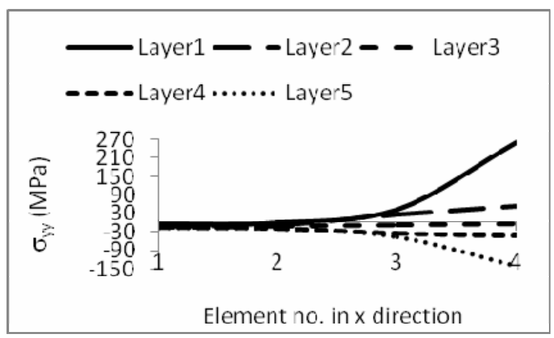


Fig 7: Stress distribution for case 3 with rectangular element (E varies exponentially).

Case 4. With thermal stress & asymmetric boundary condition

It is found that the stresses and strains vary most significantly for case 4, from element to element, for both linear and exponential cases. But the numerical values are different for the triangular and rectangular cases. Some of the results are given in Figs. 8,9.

Since the triangular element assumes constant strain in the element, the rectangular element provides a more reliable result. In general, it is concluded that whenever asymmetric boundary conditions or thermal stresses are present, every element will undergo different values of stresses. Since perfectly symmetric boundary conditions are difficult to maintain and also temperature change is unavoidable, so, the elastic body made of FGM will have variable stresses from point to point as shown in Figs. 7-9.

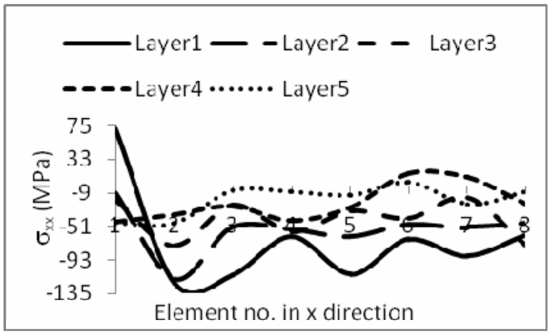


Fig 8: Stress distribution for case 4 with triangular element (E varies linearly).

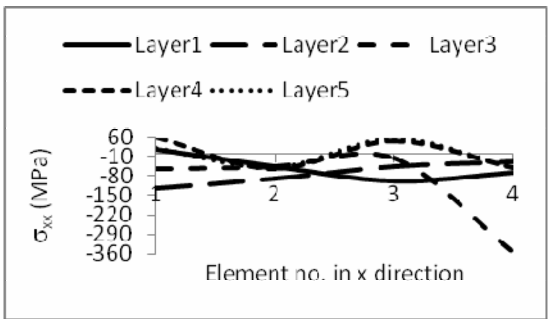


Fig 9: Stress distribution for case 4 with rectangular element (E varies linearly).

4. CONCLUSIONS

This study reveals that the effect of thermal stress and asymmetric boundary conditions on a 2D rectangular FGM elastic body is to cause variable stresses within the body. All the three components of plane stress will vary from point to point in the body if anyone of the above factors are present. But when thermal stress and asymmetric boundary conditions are both absent, the stresses are constant throughout the body.

In this work, a regular shaped body was analyzed. Since FEM is even better suited for irregular shapes, future researches can focus on irregular FGM bodies to find out the effect of thermal stress and boundary conditions.

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6. NOMENCLATURE

Symbol	Meaning	Unit
E:	Modulus of elasticity	GPa
x:	Axial distance	mm
ε:	strain	
σ:	Stress	MPa