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ANALYSIS OF ELASTIC FIELD IN A SHORT COLUMN OF ORTHOTROPIC COMPOSITE BY FINITE DIFFERENCE TECHNIQUE

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ABSTRACT
In this paper, the elastic field in a short column of orthotropic composite material under a linearly varying distributed load is analyzed. The mixed boundary value problem is formulated in terms of a single displacement potential function. By discretizing the governing equation as well as the boundary conditions, the problem is reduced to the solution of a set of simultaneous linear algebraic equations. To present numerical results, a boron/epoxy short column is considered and solution of different components of stress and strain at different sections of the column is obtained by L-U decomposition method. Both the qualitative and quantitative results establish that the single displacement potential function formulation is relatively simple and superior to analyze the elastic field in orthotropic composite structures under mixed boundary conditions.

Keywords: Displacement potential function, Finite difference method, Orthotropic composite, Short column, Mixed boundary conditions, Elastic field.
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1. INTRODUCTION

Short columns and compression members are used in different engineering machinery and structures. As orthotropic composite materials have higher strength-weight ratio than its corresponding isotropic materials, their demand in the parts of engineering machinery and structures is gradually increasing. So, understanding of elastic behavior of these structural elements of composite materials is necessary to design them efficiently.

Among the existing mathematical models of plane boundary-value stress problems, the stress function approach [1] is noticeable. Conway and Ithaca [2] modified the stress function formulation in the form of Fourier integrals for orthotropic materials, and obtained analytical solutions for a number of ideal problems considering stress boundary conditions. Displacement formulation is another well known approach in solving plane elastic problems. However, the displacement formulation involves finding two displacement functions simultaneously from the two second-order elliptic partial differential equations of equilibrium, which is extremely difficult, and this problem becomes more serious when the boundary conditions are mixed [3]. To overcome the problems involved with the mixed boundary conditions, the displacement potential formulation of isotropic material was developed and several problems have been solved both analytically and numerically [4-6]. The superiority of displacement potential approach over the

existing elasticity formulations has been established in these references. Recently, Nath *et al*. developed displacement potential formulations for orthotropic composite materials and solved a number of mixed boundary problems of structural elements of these materials [7-11]. The present study focuses on the application of displacement potential approach to the solution of elastic field in a short column of orthotropic composite under mixed boundary conditions. The problem is formulated in terms of a single potential function. Using a finite difference scheme, the governing equation and the boundary conditions are discretized which are solved by L-U decomposition method. Some numerical results of stress and displacement with reference to a boron/epoxy orthotropic column are presented graphically.

2. NUMERICAL MODEL OF THE PROBLEM

In this study, a short column of orthotropic composite material under linearly varying load as shown in Fig. 1 is considered. The supporting edge of the column is rigidly fixed while the tip of the column is subjected to a linearly varying compressive load. The relevant boundary conditions are given in Table 1.
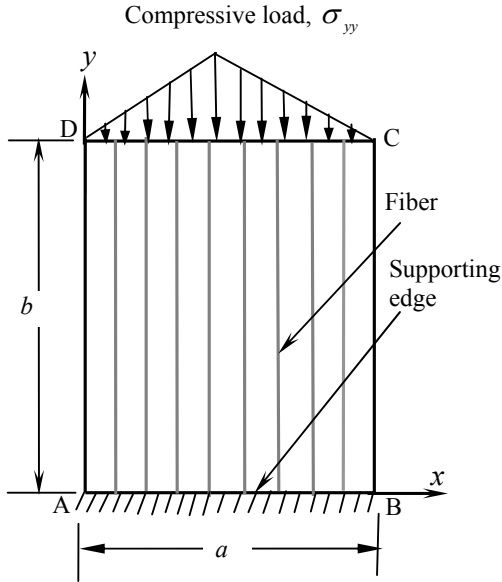
Fig.1: A short column of orthotropic composite under linearly varying compressive load.

3. BASIC FORMULATION FOR ORTHOTROPIC COMPOSITE MATERIALS

With reference to a rectangular Cartesian coordinate system (x, y), the equations of equilibrium, for the case of two-dimensional problems and in the absence of body forces, are [1]

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = 0 \tag{1}$$

$$\frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{xy}}{\partial x} = 0 \tag{2}$$

The condition of compatibility is

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} \tag{3}$$

Substituting the stress-strain and reciprocal relations of orthotropic composite materials [12], Eqs.(1) and (2) can be reduced to

$$\left(\frac{E_1^2}{E_1 - v_{12}^2 E_2} \right) \frac{\partial^2 u_x}{\partial x^2} + \left(\frac{v_{12} E_1 E_2}{E_1 - v_{12}^2 E_2} + G_{12} \right) \frac{\partial^2 u_y}{\partial x \partial y}$$
$$+ G_{12} \frac{\partial^2 u_x}{\partial y^2} = 0 \tag{4}$$

$$\left(\frac{E_1 E_2}{E_1 - v_{12}^2 E_2} \right) \frac{\partial^2 u_y}{\partial y^2} + \left(\frac{v_{12} E_1 E_2}{E_1 - v_{12}^2 E_2} + G_{12} \right)$$
$$\frac{\partial^2 u_x}{\partial x \partial y} + G_{12} \frac{\partial^2 u_y}{\partial x^2} = 0 \tag{5}$$

The compatibility condition given by Eq.(3) reduces to

$$\left(\frac{1}{E_1} \frac{\partial^2 \sigma_{xx}}{\partial y^2} + \frac{1}{E_2} \frac{\partial^2 \sigma_{yy}}{\partial x^2} \right) - \frac{v_{12}}{E_1}$$
$$\left(\frac{\partial^2 \sigma_{yy}}{\partial y^2} + \frac{1}{E_2} \frac{\partial^2 \sigma_{xx}}{\partial x^2} \right) = \frac{1}{G_{12}} \frac{\partial^2 \sigma_{xy}}{\partial x \partial y} \tag{6}$$

4. DISPLACEMENT POTENTIAL FORMULATION

The two displacement components u_x and u_y are defined in terms of a displacement potential function ψ as

$$u_x(x, y) = \frac{\partial^2 \psi}{\partial x \partial y} \tag{7}$$

$$u_y(x, y) = -\frac{1}{Z_{11}} \left[E_2^2 \frac{\partial^2 \psi}{\partial x^2} + G_{12} \left(E_2 - v_{12}^2 E_1 \right) \frac{\partial^2 \psi}{\partial y^2} \right] \tag{8}$$

where $Z_{11} = v_{12} E_1 E_2 + G_{12} \left(E_1 - v_{12}^2 E_2 \right)$.

When this definition of displacement potential function is used in Eqs.(4) and (5), Eq.(4) is automatically satisfied and Eq.(5) turns into the following form

$$E_1 G_{12} \frac{\partial^4 \psi}{\partial x^4} + E_2 \left(E_1 - 2 v_{12} G_{12} \right) \frac{\partial^4 \psi}{\partial x^2 \partial y^2} + E_2 G_{12} \frac{\partial^4 \psi}{\partial y^4} = 0 \tag{9}$$

Making use of Eqs. (7), (8), the stress-strain and reciprocal relations, the three stress components can be expressed in terms of displacement potential function ψ as

$$\sigma_{xx}(x, y) = \frac{E_2 G_{12}}{Z_{11}} \left[E_2 \frac{\partial^3 \psi}{\partial x^2 \partial y} - v_{12} E_1 \frac{\partial^3 \psi}{\partial y^3} \right] \tag{10}$$

$$\sigma_{yy}(x, y) = \frac{E_1 E_2}{Z_{11}} \left[\left(v_{12} G_{12} - E_2 \right) \frac{\partial^3 \psi}{\partial x^2 \partial y} - G_{12} \frac{\partial^3 \psi}{\partial y^3} \right] \tag{11}$$

$$\sigma_{xy}(x, y) = -\frac{E_2 G_{12}}{Z_{11}} \left[E_2 \frac{\partial^3 \psi}{\partial x^3} - v_{12} E_1 \frac{\partial^3 \psi}{\partial x \partial y^2} \right] \tag{12}$$

5. FINITE DIFFERENCE DISCRETIZATION

Using the central difference scheme, Eq. (9) can be expressed as

$$zk1\{\psi(i-2,j)+\psi(i+2,j)\}-zk2\{\psi(i-1,j)+$$
$$\psi(i+1,j)\}-zk3\{\psi(i,j+1)+\psi(i,j-1)\}+$$
$$zk4\psi(i,j)+zk5\{\psi(i+1,j+1)+\psi(i+1,j-1) \quad (13)$$
$$+\psi(i-1,j+1)+\psi(i-1,j-1)\}+Q\{\psi(i,j-2)$$
$$+\psi(i,j+2)\}=0$$

The stencil of Eq. (13) and the grid structure of the column is shown in Fig. 2. To avoid the inclusion of additional points outside the imaginary boundary, different discretized forms of the same boundary condition is used in different boundary segments of the column. The different discretized forms include i-forward and j-forward; i-backward and j-forward; i-backward and j-backward; i-forward and j-backward. As an example, only one particular form of each boundary condition is given here as

$$u_x(i,j)=c_1[9\psi(i,j)-12\{\psi(i,j+1)+\psi(i+1,j)\}+$$
$$16\psi(i+1,j+1)+3\{\psi(i,j+2)+\psi(i+2,j)\}- \quad (14)$$
$$4\{\psi(i+1,j+2)+\psi(i+2,j+1)\}+\psi(i+2,j+2)]$$

$$u_y(i,j)=c_2\{\psi(i-1,j)+\psi(i+1,j)\}+c_3\{\psi(i,j+1) \quad (15)$$
$$+\psi(i,j-1)\}-2(c_2+c_3)\psi(i,j)\}$$

$$\sigma_{xx}(i,j)=-3c_4\{\psi(i-1,j)+\psi(i+1,j)\}+$$
$$4c_4\{\psi(i-1,j+1)+\psi(i+1,j+1)\}$$
$$+(6c_4+10c_5)\psi(i,j)-(8c_4+12c_5)\psi(i,j+1)+ \quad (16)$$
$$(2c_4+6c_5)\psi(i,j+2)-c_4\psi(i-1,j+2)$$
$$-c_4\psi(i+1,j+2)-3c_5\psi(i,j-1)-c_5\psi(i,j+3)$$

$$\sigma_{yy}(i,j)=-3c_6\{\psi(i-1,j)+3c_6\psi(i+1,j)\}+$$
$$4c_6\{\psi(i-1,j+1)+\psi(i+1,j+1)\}$$
$$+(6c_6+10c_7)\psi(i,j)-(8c_6+12c_7)\psi(i,j+1)+ \quad (17)$$
$$(2c_6+6c_7)\psi(i,j+2)-c_6\psi(i-1,j+2)-$$
$$c_6\psi(i+1,j+2)-3c_7\psi(i,j-1)-c_7\psi(i,j+3)$$

$$\sigma_{xy}(i,j)=-3c_9\psi(i-1,j)+(10c_9+6c_8)\psi(i,j)-$$
$$(12c_9+8c_8)\psi(i+1,j)+(2c_8+6c_9)$$
$$\psi(i+2,j)-c_9\psi(i+3,j)-3c_8\{\psi(i,j-1)+ \quad (18)$$
$$\psi(i,j+1)\}+4c_8\{\psi(i+1,j-1)+$$
$$\psi(i+1,j+1)\}-c_8\{\psi(i+2,j-1)+\psi(i+2,j+1)\}]$$

where $zk1$, $zk2$, $zk3$, $zk4$, $c_1...c_9$ are constants. The other forms of the boundary conditions can be found in the Ref. [10]. The stencils of Eqs. (14)-(18) are shown in Fig. 2.

6. RESULTS AND DISCUSSION

In this section, some numerical results are presented for a column of boron/epoxy orthotropic composite.
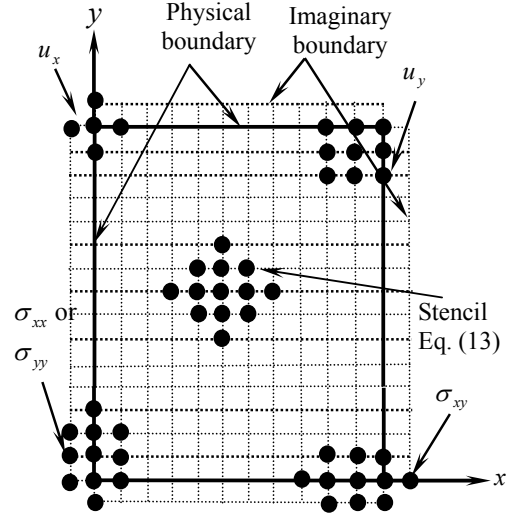

Fig.2 Stencils of the governing equation and boundary conditions.

Although the formulations can be applied to any composite, the boron/epoxy composite is chosen merely as an example. The effective mechanical properties of the boron/epoxy composite are: $E_1 = 29.59\times10^3$ MPa, $E_2 = 2.683\times10^3$ MPa, $\nu_{12} = 0.23$, and $G_{12} = 0.81\times10^3$ MPa. Furthermore, the aspect ratio of the column used in obtaining the results is taken as $b/a = 2.0$. The maximum value of the applied stress σ_{yy}^0 is taken as 41 MPa. The domain of the column is discretized by using uniform rectangular meshes with equal dimensions in the x- and y-directions. The numbers of meshes used in obtaining the results are 36 and 19 in the y- and x-directions, respectively. The convergence of the results is verified by varying the number of meshes.
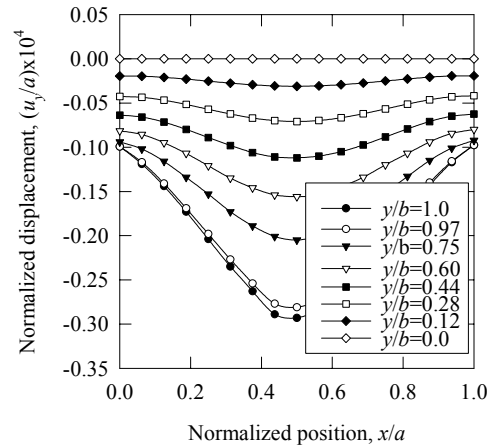

Fig 3: Normalized displacement component, u_y / a at different sections of the column.
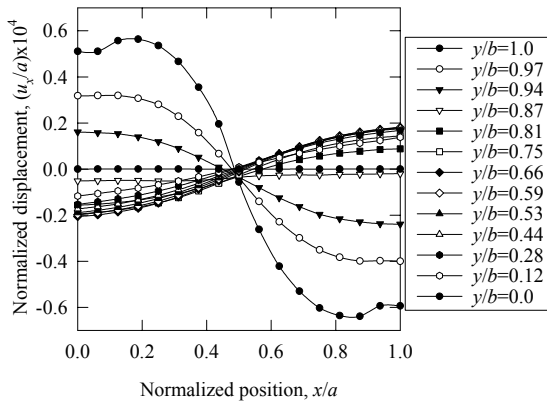
of the problem.

Fig 4: Normalized displacement component, u_x / a at different sections of the column.

The distribution of the normalized displacement component u_y / a with respect to x is presented in Fig. 3. The distribution of u_y is observed to be in good agreement with the physical characteristics of the column. As the load is applied on the top edge y/b=1.0, the displacement will be maximum there and it will gradually decrease towards the supporting edge of the column. This phenomenon is readily reflected as can be seen from the Fig. 3. Further, u_y is zero at y/b=0 which satisfies the boundary condition.

Figure 4 shows the normalized displacement component u_x / a at different sections of the column. From this figure, it is noted that the distribution is antisymmetric with respect to the vertical section x/a=0.5 of the column. In few sections near the loaded boundary of the column, the displacement u_x varies from the positive value to the negative value. For rest of the sections, the displacement u_x varies from the negative to the positive value. At the mid vertical plane x/a=0.5, u_x is zero which conforms to the physical characteristic
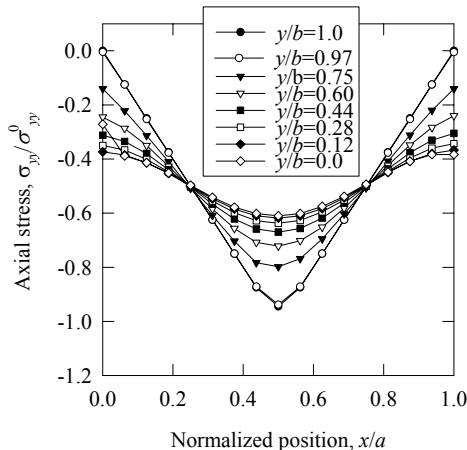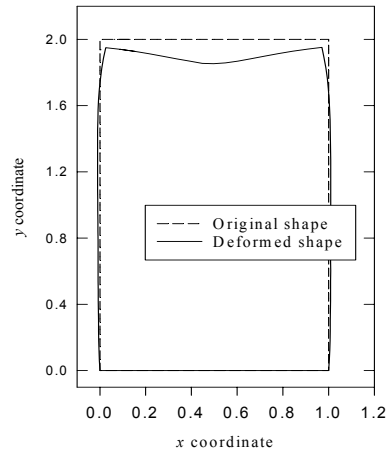

Fig 5: Deformed shape of the column (Deformation is magnified by a multiplication factor of 500).

Figure 5 shows the deformed shape of the column. The displacement u_y is seen to be more significant than the displacement u_x. Figure 6 illustrates the normalized normal stress component $\sigma_{yy} / \sigma_{yy}^0$ at different sections of the column. The normal stress component σ_{yy} is higher at the top edge and lower at the supporting edge, which is also in good agreement with the loading conditions of the problem. All the distributions are negative throughout the height of the column, which in turn indicates that the column is in compression in the vertical direction. It is noted that the variation of stress is symmetric around the vertical midsection of the column.

Figure 7 exhibits the distribution of normalized normal stress component $\sigma_{xx} / \sigma_{yy}^0$ at different sections of the column. At the loaded boundary, the stress is
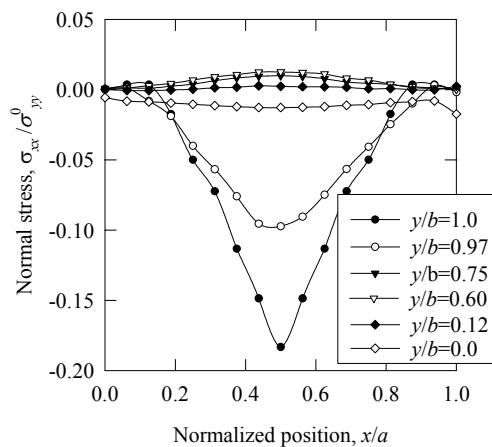

Fig 6: Normalized normal stress component, $\sigma_{yy} / \sigma_{yy}^0$ at different sections of the column.

Fig 7: Normalized normal stress component, $\sigma_{xx} / \sigma_{yy}^0$ at different sections of the column.
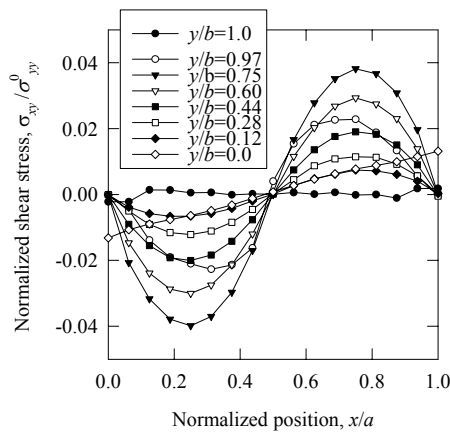
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Fig 8: Normalized shear stress component, $\sigma_{xy}/\sigma_{yy}^0$ at different sections of the column.

triangularly varied and of higher magnitude. Towards the supporting edge, this stress decreases. At the loaded boundary, the non-linearity of the distribution is higher and towards the fixed edge the non-linearity decreases and becomes almost linear according to the Sanit Venants' principle. It is noted that the maximum normal stress σ_{xx} at the loaded boundary is less than the approximately one fourth of the maximum value of the applied stress on the boundary.

Figure 8 illustrates the normalized shear stress distribution $\sigma_{xy}/\sigma_{yy}^0$ with respect to x/a at different sections of the column. It is seen that its value is maximum at the section y/b=0.75 and zero at the loaded boundary y/b=1.0. At the supporting edge, the shear stress is not zero rather it is of intermediate value as compared to those of other sections. For the sections below the top edge, the general trend of the distribution tells that the magnitude of shear stress increases gradually up to y/b=0.75 and then gradually decreases towards the supporting edge.

7. CONCLUSIONS

The problem of a short column of orthotropic composite material is formulated in terms of a single displacement potential function to determine the elastic filed for mixed boundary conditions. Using a finite difference scheme, the governing equation and the boundary conditions are discretized. For a linearly varying load at the tip of a boron/epoxy orthotropic composite column, numerical results of different components of stress and displacement are obtained. Both the qualitative and quantitative results justify the superior capability of the displacement potential approach in analyzing elastic field in structural elements of orthotropic composite material under mixed boundary conditions.

8. REFERENCES

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Table 1 Boundary conditions of the column

Boundary segments	Boundary conditions	
AB	$u_x = 0$	$u_y = 0$
BC	$\sigma_{xx} = 0$	$\sigma_{xy} = 0$
CD	$\sigma_{yy} = 2\dfrac{\sigma_{yy}^0 y}{a}$ for $x = 0$ to $\dfrac{a}{2}$ $\sigma_{yy} = -2\dfrac{\sigma_{yy}^0 y}{a} + 2\sigma_{yy}^0$ for $x = \dfrac{a}{2}$ to a	$\sigma_{xy} = 0$
DA	$\sigma_{xx} = 0$	$\sigma_{xy} = 0$

9. NOMENCLATURE

Symbol	Meaning	Unit
u_x	Displacement in x direction	(m)
u_y	Displacement in y direction	(m)
σ_{xx}	Normal stress in x direction	(MPa)
σ_{yy}	Normal stress in y direction	(MPa)
σ_{xy}	Shear stress component	(MPa)
σ_{yy}^0	Maximum value of applied stress	(MPa)
E_1	Longitudinal elastic modulus	(GPa)
E_2	Transverse elastic modulus	(GPa)
ν_{12}	Major Poisson's ratio	-