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VARIATIONAL FORMULATION BASED POST-ELASTIC ANALYSIS OF ROTATING DISKS WITH VARYING THICKNESS

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ABSTRACT
Numerical solutions under plane stress conditions for elasto-plastic deformation and stress states of rotating solid disks with variable thickness have been reported. The problem is formulated through a Variational method. Assuming a series solution and using Galerkin's principle, the solution of the governing partial differential equation is obtained. The formulation is based on von Mises yield criterion and linear strain hardening material behavior. The approximate solution is obtained using an iterative method. Results are validated with benchmark solutions.

Keywords: Variational method, von Mises, Plastic front, Limit angular speed.

1. INTRODUCTION
Due to widespread applications, the analysis of rotating disk behavior has been of great interest to many researchers. Assuming linear strain hardening material behavior, Gamer [1-2] reported the elasto-plastic behavior of rotating disk of constant thickness using Tresca's yield criterion and its associated flow rule. Güven [3-4] carried out the analysis of rotating solid disk with variable thickness up to fully plastic state for linearly strain hardening material behavior. Based on both Tresca's and von-Mises yield criterion, Rees [5] investigated the elasto-plastic behavior of rotating disks of uniform thickness made of elastic-perfectly plastic material and reported the comparative results. You and Zhang [6] presented an approximate analytical solution for rotating solid disk of uniform thickness with plane stress assumption for non-linear strain hardening material behavior based on von-Mises yield criterion, deformation theory of plasticity and a polynomial stress-strain relationship. Based on similar assumptions, You et al. [7] developed a unified numerical method to report the elasto-plastic behavior of rotating disk of varying thickness made up of non-linearly strain hardening material. A unified yield criterion was proposed by Ma et al. [8] to report the plastic limit angular velocity and stress distribution in fully plastic state of rotating disks with variable thickness.

Eraslan and Orcan [9] obtained an analytical solution of elasto-plastic deformation field of rotating solid disks with exponentially varying thickness using Tresca's yield criterion and its associated flow rule for linear strain hardening material behavior. Eraslan and Orcan [10] extended the study for disks where, yielding initiates not at the centre but at an intermediate radial location. Eraslan [11] studied the inelastic behavior of variable thickness rotating solid disks with linear strain hardening material behavior using both Tresca's and von Mises yield criterion.

Elasto-plastic behavior of rotating disks of variable thickness in power function form for both linear and non-linear strain hardening material behavior had been studied by Eraslan and Argeso [12] using von Mises yield criterion. In another research paper [13], Eraslan carried out the elasto-plastic analysis of rotating solid and annular disks with elliptical thickness variation and made of linearly hardening material using both Tresca and von-Mises criteria. The application of variational method, proposed by Bhowmick et al. [14] has yielded a generalized approach to study the behavior of rotating solid disks of variable thickness in the elastic regime. In another paper, the method has been further extended into elasto-plastic domain [15].

In the present study, a numerical method based on variational principle for elasto-plastic analysis of rotating variable thickness disks using von Mises yield criterion has been proposed. A solution algorithm has been developed to obtain an approximate solution of the unknown displacement field from the governing set of equations in an iterative manner.

2. MATHEMATICAL FORMULATION
The centrifugal loading on a rotating disk produces radial and tangential strain field, which in turn produces stresses. The present analysis is carried out based on the assumptions that material of the disk is isotropic and homogeneous and a state of plane stress exists in the loaded condition. Stress-strain relation of the disk is

linear elastic followed by linear strain hardening as shown in Fig. 1. In case of two-dimensional stress, the general condition of yielding based on von-Mises theory is given by, $\sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2 = \sigma_y^2$, where σ_y is the uniaxial yield stress value of the disk material and σ_{vm} is the von-Mises stress. At a certain speed, known as elastic limit angular speed (Ω_1), the stress field of the disk exceeds the yield limit value, thus giving rise to a plastic front. On further increase in rotational speed, a certain region of the disk attains post-elastic state and when this region encompasses the entire disk we get plastic limit speed or collapse speed (Ω_2). The present method captures the location of plastic front numerically by using an iterative method.

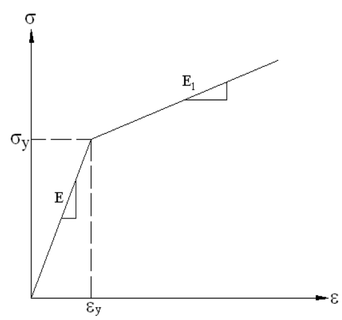


Fig 1. Linear elastic linear strain hardening material behavior.

For a uniform thickness disk the yielding initiates at the root. As a result the disks are made thicker near the hub and the thickness is reduced towards the periphery which results in variable thickness disks having higher elastic limit angular speeds. On varying the geometry it is observed that maximum stresses at elastic limit angular speed (ω_1) may occur at locations other than the root of the disk. Under such a situation, initiation of yielding occurs at location away from the root ($r = r_y$).With increasing angular speed, the centrifugal load on the disk increases and the plastic front originated at $r = r_y$ starts to spread in two directions; towards the root as well as the periphery. At such load increment ($\omega = \omega_1 + \Delta\omega$), the plastic region spans between $r = r_i$ to $r = r_o$. Further load increment gives rise to two possibilities depending on the geometry parameters. Firstly the plastic region may collapse at the root before reaching the periphery or secondly the plastic region may reach the periphery before reaching the root. The angular speed at which the plastic front reaches either root or periphery is designated as ω_1^i and ω_1^o respectively. Additional load increments cause the disk to attain a fully plastic state and the corresponding plastic limit angular speed is designated as ω_2.

Variational principle based on minimization of total potential energy functional states that

$$\delta(U + V) = 0 \tag{1}$$

where, $U = U^i{}_e + U_p + U^o{}_e$, i.e., total strain energy, U consists of two elastic $\{(U^i{}_e)$ and $(U^o{}_e)\}$ and a plastic (U_p) part, V is the potential of the external forces and δ is the variational operator . The interface between the inner elastic and the intermediate plastic region is demarcated by the radius $r = r_i$ and that between the plastic region and the outer elastic region is demarcated by the radius $r = r_o$. The variation of elastic part of strain energy (for $r = a \ to \ r = r_i$) is given as,

$$\delta\left(U^i{}_e\right) = \frac{2\pi E}{1-\mu^2}\int_a^{r_i}\left\{\frac{u\delta u}{r} + \nu u\delta\left(\frac{du}{dr}\right) + \nu\frac{du}{dr}\delta u + r\frac{du}{dr}\delta\left(\frac{du}{dr}\right)\right\}h\,dr \tag{2}$$

Similarly the variation of elastic part of strain energy (for $r = r_o \ to \ r = b$) is given as,

$$\delta\left(U^o{}_e\right) = \frac{2\pi E}{1-\mu^2}\int_{r_o}^{b}\left\{\frac{u\delta u}{r} + \nu u\delta\left(\frac{du}{dr}\right) + \nu\frac{du}{dr}\delta u + r\frac{du}{dr}\delta\left(\frac{du}{dr}\right)\right\}h\,dr \tag{3}$$

Based on Hencky's deformation theory of plasticity the variation of post elastic part of strain energy (for $r = r_i$ to $r = r_0$) is given by,

$$\delta U_p = \delta\left\{\int_{vol}\left(dU_r + dU_t\right)dv\right\} \tag{4}$$

where, dU_r and dU_t are the contributions coming from radial and tangential stresses and strains and are given as

$$dU_r = \frac{1}{2}\sigma_r{}^0\varepsilon_r{}^0 + \frac{1}{2}\sigma_r{}^p\varepsilon_r{}^p + \sigma_r{}^0\varepsilon_r{}^p \tag{5a}$$

$$dU_t = \frac{1}{2}\sigma_t{}^0\varepsilon_t{}^0 + \frac{1}{2}\sigma_t{}^p\varepsilon_t{}^p + \sigma_t{}^0\varepsilon_t{}^p \tag{5b}$$

From the stress-strain compatibility relations and strain-displacement relations it is known that

$$\sigma_r{}^0 = \frac{E}{(1-\nu^2)}\left[\varepsilon_r{}^0 + \nu\varepsilon_t{}^0\right], \sigma_t{}^0 = \frac{E}{(1-\nu^2)}\left[\varepsilon_t{}^0 + \nu\varepsilon_r{}^0\right]$$

$$\sigma_r{}^p = \frac{E_1}{(1-\nu^2)}\left[\varepsilon_r{}^p + \nu\varepsilon_t{}^p\right], \sigma_t{}^p = \frac{E_1}{(1-\nu^2)}\left[\varepsilon_t{}^p + \nu\varepsilon_r{}^p\right]$$

$\varepsilon_r{}^p = \frac{du}{dr} - \varepsilon_r{}^0, \varepsilon_t{}^p = \frac{u}{r} - \varepsilon_t{}^0$, where the super-script ^{0} refers to the state of initiation of yielding,

Substituting the abovementioned relations in the expressions of dU_r and dU_t the final expression for δU_p is obtained as

$$\delta U_p = \frac{E_1}{\left(1-\nu^2\right)}\int_{r_i}^{r_o}\left\{\begin{aligned}&\frac{u\delta u}{r}+r\left(\frac{du}{dr}\right)\delta\left(\frac{du}{dr}\right)\\&+\nu\left(\frac{du}{dr}\delta u+u\delta\left(\frac{du}{dr}\right)\right)\end{aligned}\right\}2\pi h dr \tag{6}$$

$$+\frac{E-E_1}{\left(1-\nu^2\right)}\int_{r_i}^{r_o}\left\{\begin{aligned}&\varepsilon_r{}^0\left(\nu\delta u+r\delta\left(\frac{du}{dr}\right)\right)\\&+\varepsilon_t{}^0\left(\delta u+\nu\delta\left(\frac{du}{dr}\right)r\right)\end{aligned}\right\}2\pi h dr$$

The variation of the potential of the external forces is given by,

$$\delta(V) = -2\pi\,\rho\,\omega^2\int_a^b r^2 h\delta(u)dr \tag{7}$$

The normalization of Eqs (2, 3, 6 and 7) is carried out with four parameters ($\Delta, \Delta_1, \Delta_2$ and Δ_3) and four normalized coordinates (ξ, ξ_1, ξ_2 and ξ_3), where $\Delta = b-a$ and $\xi = (r-a)/\Delta$, $\Delta_1 = r_i - a$ and $\xi_1 = (r-a)/\Delta_1$, $\Delta_2 = r_0 - r_i$ and $\xi_2 = (r-r_i)/\Delta_2$, $\Delta_3 = b - r_0$ and $\xi_3 = (r-r_o)/\Delta_2$

The global displacement field $u(\xi)$ is approximated by co-ordinate functions $u(\xi) \cong \sum c_i\phi_i$, i=1, 2,…, n, where ϕ_i is the set of orthogonal functions developed through Gram-Schmidt orthogonalization scheme. The necessary starting function to generate the higher order orthogonal functions is selected by satisfying the relevant boundary conditions ($u\big|_{r=0} = 0$ and $\sigma_r\big|_{r=b} = 0$), of a rotating solid disk in elastic regime. For an annular disk boundary conditions become ($\sigma_r\big|_{r=a} = 0$ and $\sigma_r\big|_{r=b} = 0$). To facilitate computation, displacement functions in the elastic and post-elastic regions are expressed as $u(\xi_1) \cong \sum c_i\phi_i^{e1}, u(\xi_2) \cong \sum c_i\phi_i^{p}$ and $u(\xi_3) \cong \sum c_i\phi_i^{e2}$ respectively.

On substituting the normalized expressions of $\delta(U^i{}_e), \delta(U_p), \delta(U^o{}_e), \delta(V),$ in Eq. (1), replacing the displacement field with assumed co-ordinate functions and replacing operator δ by $\partial/\partial c_j$, the governing set of equations is obtained in matrix form as follows:

$$\frac{E}{1-\mu^2}\sum_{j=1}^{n}\sum_{i=1}^{n}c_i\int_0^1\left\{\frac{\phi_i^{e1}\phi_j^{e1}}{(\Delta_1\xi_1+a)}+\frac{(\Delta_1\xi_1+a)}{(\Delta_1)^2}\phi_i^{e1'}\phi_j^{e1'} + \frac{\nu}{\Delta_1}(\phi_i^{e1'}\phi_j^{e1}+\phi_i^{e1}\phi_j^{e1'})\right\}h\Delta_1 d\xi_1$$

$$+\frac{E_1}{\left(1-\nu^2\right)}\sum_{j=1}^{n}\sum_{i=1}^{n}c_i\int_0^1\left\{\frac{(\Delta_2\xi_2+r_i)}{(\Delta_2)^2}\phi_i^{p'}\phi_j^{p'}+\frac{\phi_i^p\phi_j^p}{(\Delta_2\xi_2+r_i)} + \frac{\nu}{\Delta_2}\left\{\phi_i^{p'}\phi_j^{p}+\phi_i^{p}\phi_j^{p'}\right\}\right\}h\Delta_2 d\xi_2$$

$$+\frac{E}{1-\mu^2}\sum_{j=1}^{n}\sum_{i=1}^{n}c_i\int_0^1\left\{\frac{\phi_i^{e2}\phi_j^{e2}}{(\Delta_3\xi_3+r_o)}+\frac{(\Delta_3\xi_3+r_o)}{(\Delta_3)^2}\phi_i^{e2'}\phi_j^{e2'} + \frac{\nu}{\Delta_3}(\phi_i^{e2'}\phi_j^{e2}+\phi_i^{e2}\phi_j^{e2'})\right\}h\Delta_3 d\xi_3$$

$$=\rho\omega^2\sum_{j=1}^{n}\int_0^1\left\{(\Delta\xi+a)^2\phi_j\right\}h\Delta d\xi$$

$$-\frac{E-E_1}{\left(1-\nu^2\right)}\int_0^1\left\{\varepsilon_r{}^0\left(\nu\phi_j^p+\frac{(\Delta_2\xi_2+r_i)}{\Delta_2}\phi_j^{p'}\right) + \varepsilon_t{}^0\left(\phi_j^p+\nu\frac{(\Delta_2\xi_2+r_i)}{\Delta_2}\phi_j^{p'}\right)\right\}h\Delta_2 d\xi_2 \tag{8}$$

In the above equation $()'$ indicates differentiation with respect to normalized coordinates. It should be noted further that he mathematical formulation is presented in generalized form, to be applicable both for solid and annular disks. However, in the present study the case of solid disks are only considered by setting $a = 0$.

2.1 Solution Algorithm

The governing Eq. (8) can be expressed in matrix form as, $[K]\{c\} = \{f\}$, where [K] is the stiffness matrix and {f} is the load vector and the required solution of unknown coefficients $\{c\}$ is obtained numerically by using an iterative scheme. A brief description of the solution algorithm is provided below.

1. Solve for limit angular speed [14] and initiate three loops.
2. Start the outer (first) loop and increase the rotational speed through a suitable step size, (Ω_l+$\Delta\Omega_l$).
3. Start intermediate (second) loop and give r_0 a small increment.
4. Start inner (third) loop and give r_i a small increment.
5. The ratio of $\sigma_t{}^0$ and $\sigma_r{}^0$ in the elastic region is constant for a particular radial location. This ratio (k) is stored during elastic analysis. Whenever yield front expands at a particular radial location, the values of $\sigma_t{}^0$ and $\sigma_r{}^0$ at that point can be obtained as follows (considering that up to yield point the ratio k is maintained):

$$\sigma_{vm} = \left(\sigma_t{}^2+\sigma_r{}^2-\sigma_t\sigma_r\right)^{\frac{1}{2}} = \sigma_r\left(1+k-k^2\right)^{\frac{1}{2}}$$

6. Obtain the displacement field using the elastic-plastic formulation of Eq. (8) and post process to check whether von Mises stress attains the value of yield stress at the yield front location.
7. If the condition in step 6 is true within the set error limit then current position is the exact yield front location and go to step 2.

8. If the condition in step 6 is not satisfied, go to step 3, refine the increment on r_0 and continue the iterative process.

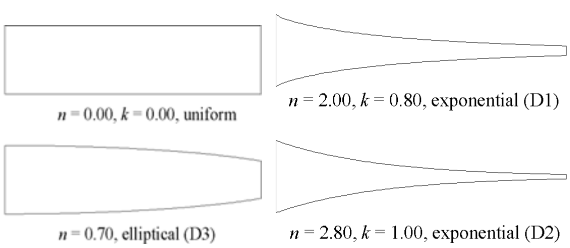
3. RESULTS AND DISCUSSIONS

The results are presented in terms of the following dimensionless and normalized variables: radial coordinates $\xi = r/b$, angular velocity $\Omega = \omega b\sqrt{\rho / \sigma_y}$, stress $\bar{\sigma} = \sigma / \sigma_y$, displacement $\bar{u} = uE / b\sigma_y$ and hardening parameter $\overline{H} = E_1 / \left(E - E_1\right)$. The results are generated using b=1.0m, ρ =7850 kg/m^3, E=210 GPa and σ_y =350 MPa. The value of poisson's ratio ν is taken as 1/3 unless otherwise stated. For the validation purpose, the value of ν is taken as 0.5 in post-elastic region. Validation of the numerical scheme is presented for disks with uniform and elliptical thickness variation whereas results are presented for disks of exponential thickness variation. The mathematical expressions for thickness variations are defined below:

$h(\xi) = h_0 \exp[-n\xi^k]$ for exponential disk,

$h(\xi) = h_0\sqrt{1 - n\xi^2}$ for elliptical variation.

Here, n, k are the parameters defining the geometry and h_0 is the thickness at the root. From the above expressions, disk of uniform thickness is obtained for n = 0.0. The disk profiles treated in the present study are shown in Fig. 2 (a-b). In Fig. 2 (a), the profile of uniform disk and elliptical disk is illustrated. In Fig. 2 (b) profile of exponential disks for different geometry parameters is shown.



(a) (b)

Fig 2. Disk profiles with geometry parameters

Table1: Validation of results on Ω_2

Geometry	Parameters	Ω_2	
		Present method	Existing results
Uniform	n=0.0	2.13726	2.11747 [11]
Elliptical	n=0.7	2.28764	2.25761[13]

Results for linear strain hardening material behavior have been generated using $\overline{H}$ =0.5. A validation on the plastic limit angular speed Ω_2 is carried out and the results are presented in Table 1. The plots for normalized displacement and normalized radial and tangential stresses at the plastic limit speed Ω_2 obtained by the present study for uniform and elliptical disk is presented in Fig.3 (a, b). In Fig.4, the propagation of elastic-plastic interface radius with angular speed is plotted for an

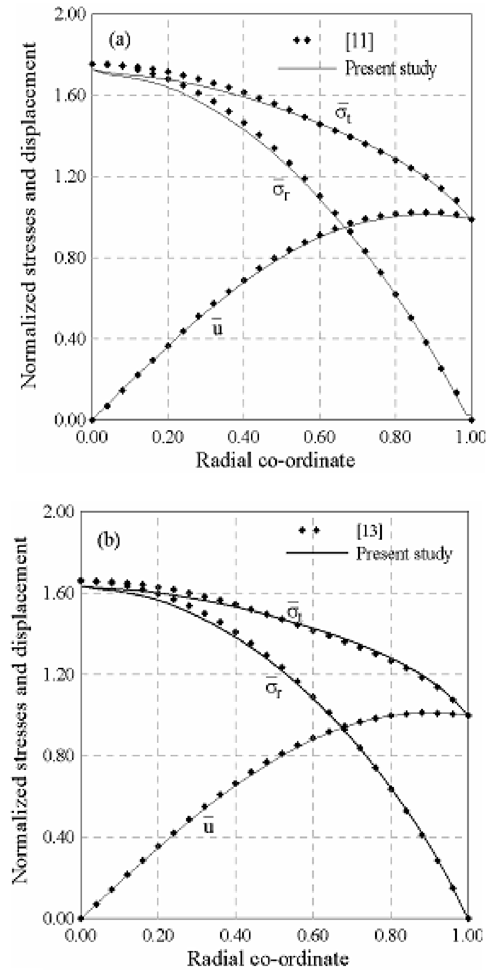


Fig 3. Validation of normalized displacement and normalized radial and tangential stresses at Ω_2 for (a) Uniform disk and (b) Elliptical disk.

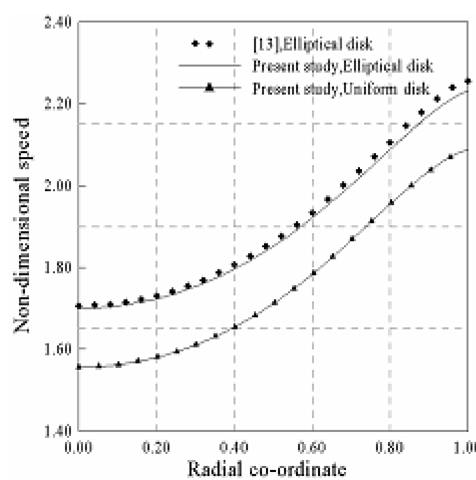


Fig 4. Propagation of elastic-plastic interface for Uniform and Elliptical disk.

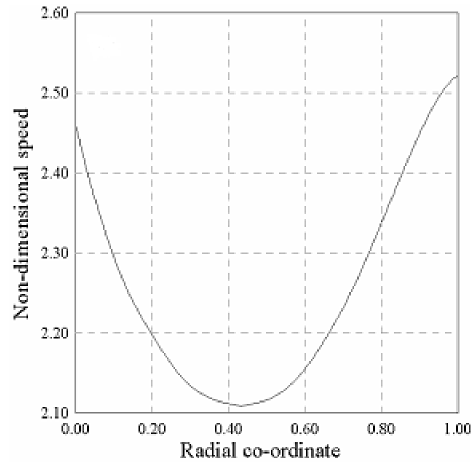


Fig 5. Propagation of elastic-plastic interface for Exponential disk (D1) at $\Omega_2 = 2.5213$

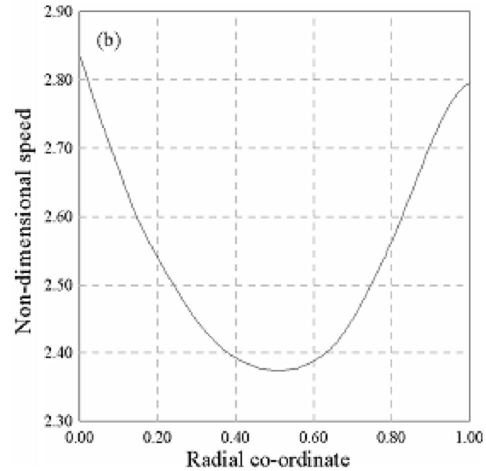


Fig 7. Propagation of elastic-plastic interface for Exponential disk (D2) at $\Omega_2 = 2.8372$

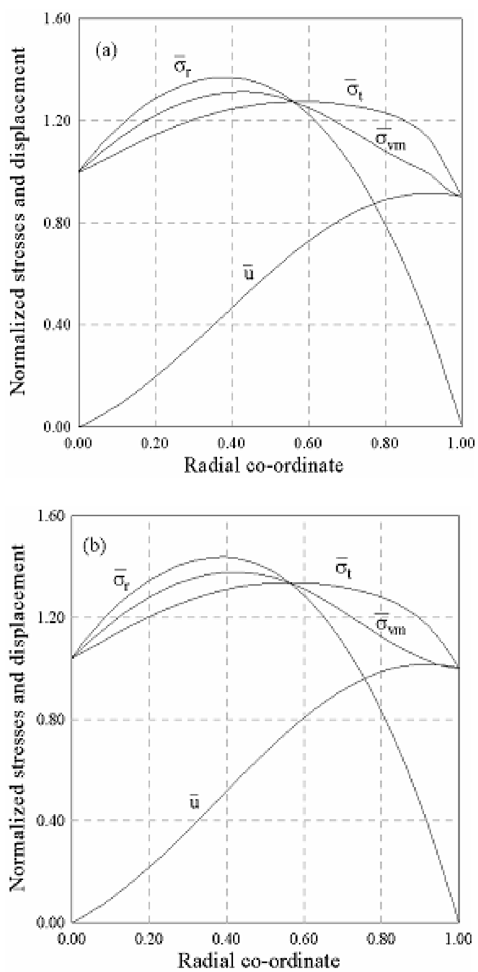


Fig 6. Normalized displacement, radial, tangential and von-Mises stresses for Exponential disk (D1) at (a) $\Omega_1^i = 2.4621$ and (b) $\Omega_2 = 2.5213$

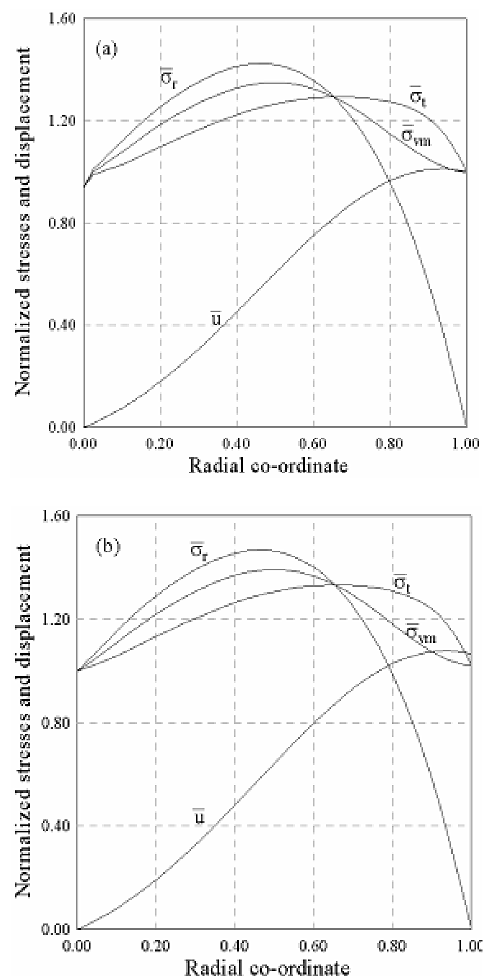


Fig 8. Normalized displacement, radial, tangential and von-Mises stresses for Exponential disk (D2) at (a) $\Omega_1^o = 2.7948$ and (b) $\Omega_2 = 2.8372$

elliptical disk and is validated with the results available in [13]. These plots exhibit very good agreement establishing validity of the present elasto-plastic analysis method.

The displacement and stress state of exponential disk is investigated next. The geometry parameters (***n*** =2.0, ***k*** =0.8 (D1) and ***n*** =2.8, ***k*** =1.0 (D2)) are selected to ensure yielding at locations away from the root. Furthermore, for disk type D1, the yield front propagates in a manner to reach root first and then the periphery, whereas for disk type D2 the yield front reaches periphery first.

For disk type D1, the dimensionless angular speed (Ω_1^i) at which the yield front reaches the root is 2.4621. The fully plastic state is attained at Ω_2 = 2.5213. In Fig.5 the propagation of elastic-plastic interface is plotted against increasing speed. The corresponding normalized radial, tangential and von Mises stresses and normalized displacement at Ω_1^i and Ω_2 for disk type D1 is plotted in Fig. 6(a, b).

For disk type D2, the dimensionless angular speed (Ω_1^o) is 2.7948 and the fully plastic state is attained at Ω_2 = 2.8372. The propagation of elastic-plastic interface is plotted against increasing speed in Fig.7 for disk type D2. The corresponding normalized radial, tangential and von Mises stresses and normalized displacement at Ω_1^o and Ω_2 is plotted in Fig. 8(a, b).

4. CONCLUSION

The present work gives an approximate solution in the elasto-plastic region of a solid rotating disk of varying thickness assuming linear strain hardening material behavior following von Mises yield criterion. The results obtained by the present methodology have been validated and it showed a close conformity with the existing results of similar problem.

Some new results for displacement field and radial, tangential and von Mises stress field at plastic limit speed for exponential disks yielding at locations away from the root for linear strain hardening material behavior have been furnished and plots showing the bi-directional advancement of the plastic front with increase in rotational speed have been presented.

The method of formulation readily gives the kernel for dynamic analysis and many other complicating effects. The results are presented graphically so that they become designer friendly. The method developed has application potential in various other problems, e.g., shrink fitted rotating disk, pre-stressed rotating disk, compound disk made from different materials, etc.

5. ACKNOWLEDGEMENT

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