Proceedings of the
International Conference on Mechanical Engineering 2009
(ICME2009) 26- 28 December 2009, Dhaka, Bangladesh

ICME09-AM-08

LARGE AMPLITUDE PERIODIC VIBRATION OF COMPOSITE CYLINDRICAL PANELS

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ABSTRACT
Large amplitude free flexural vibration characteristics of isotropic and multi-layered composite cylindrical shell panels are studied here using the finite element method. Formulation is based on first order shear deformation theory incorporating geometric nonlinearity. Periodic solution in time is assumed and the Galerkin's method is applied on the nonlinear governing equation to generate the matrix amplitude equation which is solved by direct iteration technique to obtain the disturbing energy *versus* vibration amplitude relationship. Dynamic response for the nonlinear free vibration of composite cylindrical panels is also obtained using Newmark's time integration technique. Specific numerical results are reported to show the effects of radius-to-span ratio, thickness-to-span ratio, and boundary condition on the large amplitude free vibration characteristics of laminated cylindrical shell panels.

Keywords: Nonlinear Vibration, Cylindrical Panels, Composite Material.

1. INTRODUCTION
Thin-walled structural components, made of isotropic or composite materials are increasingly used in high performance engineering applications and are subjected to hostile environment, resulting in a strong need to understand their nonlinear dynamic behaviour. Extensive literature review on the geometrically nonlinear flexural vibration characteristics of circular cylindrical shells and shell panels is given by Amabili and Paıdoussis [1]. The flexural vibration characteristics of shell type of structures are complex due to the asymmetric oscillation with respect to the un-deformed middle surface. The mid-surface gets compressed while deflecting inward (towards the center of curvature), while the mid-surface is subjected to tensile in-plane stress when deflecting outwards. It is observed from the existing literature, that the assumed space-mode analytical methods are widely used by different investigators while studying the geometrically nonlinear flexural vibration characteristics of curved panels [2-5]. Extensive theoretical and experimental studies on the large amplitude flexural vibration characteristics of cylindrical and doubly-curved panels are reported by Amabili [6-8]. The nonlinear strain-displacement relationships from Donnell's shell theory and Novozhilo's shell theory were employed. A multi-mode expansion with assumed approximate time functions was employed to obtain the nonlinear governing equation with finite degrees of freedom, which was solved by arc-length continuation method to investigate the flexural vibration characteristics of such panels.

Numerical techniques, such as finite element method, overcome the limitations of assumed space-mode. However, the selection of appropriate time function and determination of a steady-state periodic solution of the differential equations with quadratic and cubic nonlinearity is a challenge to the researchers working in the area of nonlinear dynamics of composite shells. Ribeiro [9-11] employed *p*-version finite element with hierarchic basis functions and harmonic balance method to get the equation of motion in frequency domain, which was solved by predictor-corrector method [9] or arc-length continuation method [10] for nonlinear free vibration or by shooting and Newton's method [11] for nonlinear forced vibration of shell panels. A strong modal interaction with even and odd harmonics was observed in the geometric nonlinear vibration, which was characterized by time plots, phase planes and Fourier spectra. Few nonlinear transient analyses of shell panels are available in the literature [12-14]. However, further study is required to understand the steady-state periodic and asymmetric vibration behavior of curved panels under transverse dynamic load.

In the present paper a sixteen nodded shell finite element [15] is employed to study the nonlinear free flexural vibration behavior of shallow composite cylindrical panels. Time function is assumed for the nonlinear equation and Galerkin's method is employed to obtain the nonlinear frequencies of vibration. Dynamic response of curved panels is also obtained by Newmark's direct time integration method. The effect of boundary condition and curvature on the large amplitude vibration characteristics of composite cylindrical panels is investigated in details.

2. FORMULATION

A sixteen noded degenerated isoparametric finite element is employed here to model a circular cylindrical shell panel of constant thickness *h*, radius *R*, axial length *L*, having plan-form width *a* and rise *H* as shown in Fig 1 (a). The sixteen-noded shell element is schematically shown in Fig 1(b). The normal to the mid-surface at node "*k*" at any time "*t*" is defined by the unit vector ${}^{t}V_n^k$. ${}^{t}V_1^k$ and ${}^{t}V_2^k$ are two unit vectors orthogonal to ${}^{t}V_n^k$.

Each node "*k*" has five degrees of freedom namely; u_1, u_2, u_3, α and β. The mid-surface displacement components u_1 and u_2 are along the unit vectors ${}^{0}V_1^k$ and ${}^{0}V_2^k$ respectively; and u_3 is the normal displacement along ${}^{0}V_n^k$. The changes in the direction cosines of the shell normal, given by V_n^k ($= {}^{t}V_n^k - {}^{0}V_n^k$) may be expressed in terms of nodal rotations (α and β) about the two vectors ${}^{0}V_1^k$ and ${}^{0}V_2^k$ as [15, 16]

$$V_n^k \;=\; -{}^{0}V_2^k\,\alpha + {}^{0}V_1^k\,\beta - \frac{1}{2}\left(\alpha^2+\beta^2\right){}^{0}V_n^k \qquad (1)$$

The linear (e_{ij}) and nonlinear (η_{ij}) components of the Green-Lagrange strain tensors $\varepsilon_{ij}^{DI} = (e_{ij} + \eta_{ij})$ may be written in terms of displacement components u_i (i = 1, 3) as

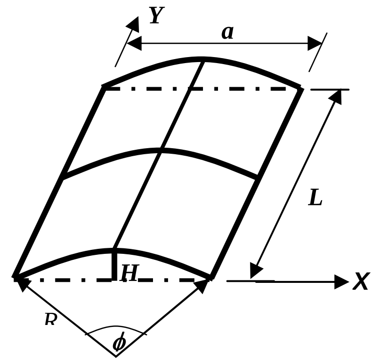
$$e_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) \qquad \text{and}$$

$$\eta_{ij} = \frac{1}{2}u_{k,i}\,u_{k,j}\,; \quad k = 1, 3 \qquad (2)$$

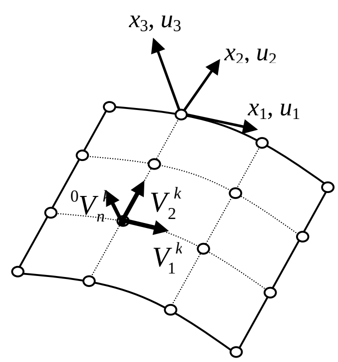
Here, the strain components are obtained by direct interpolation using the finite element displacement assumptions. The coordinate system is defined element-wise by the element isoparametric coordinates. The normal strain component ε_{33} is zero. To avoid the shear and membrane locking phenomenon, mixed-interpolated elements are constructed by using the assumed strain-fields $\boldsymbol{\varepsilon_{ij}^{AS}}$ in place of ε_{ij}^{DI}. The covariant strain components $\boldsymbol{\varepsilon_{ij}^{AS}}$ are defined as

$$\varepsilon_{ij}^{AS}(r,s) \;=\; \sum_{k=1}^{n_{ij}} N_k^{ij}(r,s)\,\varepsilon_{ij}^{DI}(r,s) \qquad (3)$$

where, the $\boldsymbol{N_k^{ij}(r,s)}$ are interpolation functions (polynomials in *r* and *s*) associated with the strain component $\boldsymbol{\varepsilon_{ij}}$ at tying point *k* and n_{ij} is the number of tying points as described in Bathe [16]. For a composite laminate of thickness *h*, comprising *N* layers with stacking angles θ_i (i = 1,2, ….., N) and layer thicknesses h_i (i = 1,2, ….., N), the necessary expressions to compute the stiffness coefficients are available in the literature [17].



(a) Geometry of cylindrical



(b) Geometry of 16 node shell

Fig 1. The geometry of a cylindrical panel and the shell element.

Following standard procedure (Lagrange's equation of motion), the nonlinear equation of equilibrium of the cylindrical shell panel under periodic transverse load may be written as

$$\begin{bmatrix} M_{mm} & 0 \\ 0 & M_{bb} \end{bmatrix}\begin{Bmatrix} \ddot{\delta}_m \\ \ddot{\delta}_b \end{Bmatrix} + \begin{bmatrix} K_{mm} & K_{mb} \\ K_{bm} & K_{bb} \end{bmatrix}\begin{Bmatrix} \delta_m \\ \delta_b \end{Bmatrix} + \begin{bmatrix} 0 & KN_1(w) \\ KN_2(w) & KN_3(w,w) \end{bmatrix}\begin{Bmatrix} \delta_m \\ \delta_b \end{Bmatrix} = \begin{Bmatrix} 0 \\ q(t) \end{Bmatrix} \qquad (4)$$

Here, ***K*** is the linear stiffness matrix; ***M*** is the mass matrix; $\boldsymbol{KN_1}$ and $\boldsymbol{KN_2}$ are the nonlinear stiffness matrices, linearly depend on transverse displacement *w*; and $\boldsymbol{KN_3}$ is the nonlinear stiffness matrix, which is a quadratic function of transverse displacement *w*. The subscripts '*m*' and '*b*' correspond to membrane (u_1, u_2) and bending (u_3, α and β) components of the degrees of freedom and corresponding mass and stiffness matrices respectively.

3. LARGE AMPLITUDE FREE VIBRATION

For the case of large amplitude vibration of shell panels, the vibration amplitude (w_{in}) towards the center of curvature is more than the corresponding amplitude in the outward direction (w_{out}), as schematically shown in

Fig 2. Hence, the displacement components in the inward (towards center of curvature) and outward direction may be individually expressed as

$$\delta_{in}(t) = \begin{Bmatrix} u_{in}\sin^2\theta_1 t,\ v_{in}\sin^2\theta_1 t,\ w_{in}\ \sin\theta_1 t, \\ (\gamma_{xz})_{in}\sin\theta_1 t,\ (\gamma_{yz})_{in}\sin\theta_1 t \end{Bmatrix}^T \tag{5}$$

$$\delta_{out}(t) = \begin{Bmatrix} u_{out}\sin^2\theta_2 t,\ v_{out}\sin^2\theta_2 t,\ w_{out} \\ \sin\theta_2 t,\ (\gamma_{xz})_{in}\sin\theta_2 t,\ (\gamma_{yz})_{in}\sin\theta_2 t \end{Bmatrix}^T \tag{6}$$

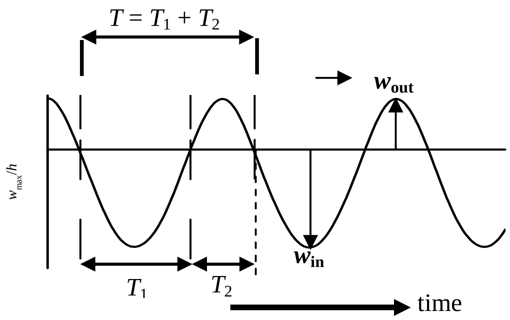


Fig 2. Schematic diagram for the large amplitude oscilation of a cylindrical shell panel.

Here, the total time period "T" is divided between the time taken "$T_1 = \pi/\theta_1$" for the inward motion and time taken "$T_2 = \pi/\theta_2$" for the outward motion (as schematically shown in Fig 2). For the case of free vibration, the strain energies $U(\delta_{in})$ and $U(\delta_{out})$ for the inward and outward vibration amplitudes should match, *i.e.*,

$$U(\delta_{in}) = U(\delta_{out}) \tag{7}$$

Substituting the assumed solutions (5, 6) into the governing equation (4) and evaluating the weighted residual along the path

$$\int_0^{T_1/2}\{R_m\}\sin^2\theta_1 t = \{0\} \text{ and } \int_0^{T_1/2}\{R_b\}\sin\theta_1 t = \{0\}$$

$$\text{or } \int_0^{T_2/2}\{R_m\}\sin^2\theta_2 t = \{0\} \text{ and } \int_0^{T_2/2}\{R_b\}\sin\theta_2 t = \{0\}$$

the following matrix-amplitude equations are obtained

$$\begin{bmatrix} \frac{3}{4}K_{mm} & \frac{8}{3\pi}K_{mb} + \frac{3}{4}KN_1 \\ \frac{8}{3\pi}K_{bm} + \frac{3}{4}KN_2 & K_{bb} + \frac{3}{4}KN_3 \end{bmatrix}\begin{Bmatrix}\delta_m \\ \delta_b\end{Bmatrix}_{in} - \theta_1^2 \begin{bmatrix} -M_{mm} & 0 \\ 0 & M_{bb}\end{bmatrix}\begin{Bmatrix}\delta_m \\ \delta_b\end{Bmatrix}_{in} = \begin{Bmatrix}0 \\ 0\end{Bmatrix} \tag{8a}$$

$$\begin{bmatrix} \frac{3}{4}K_{mm} & \frac{8}{3\pi}K_{mb} + \frac{3}{4}KN_1 \\ \frac{8}{3\pi}K_{bm} + \frac{3}{4}KN_2 & K_{bb} + \frac{3}{4}KN_3 \end{bmatrix}\begin{Bmatrix}\delta_m \\ \delta_b\end{Bmatrix}_{out} - \theta_2^2 \begin{bmatrix} -M_{mm} & 0 \\ 0 & M_{bb}\end{bmatrix}\begin{Bmatrix}\delta_m \\ \delta_b\end{Bmatrix}_{out} = \begin{Bmatrix}0 \\ 0\end{Bmatrix} \tag{8b}$$

The matrix amplitude equations (8a, b) are solved separately to get the vibration mode shapes $\{\delta_m, \delta_b\}_{in}^T$ or $\{\delta_m, \delta_b\}_{out}^T$ corresponding to inward (w_{in}) and outward (w_{out}) vibration amplitudes. Thereafter, the vibration amplitudes (w_{in} and w_{out}) are adjusted iteratively to satisfy equation (7). Further, a time history analysis is carried out starting from the initial condition $\{\delta_m, \delta_b\}_{out}^T$, obtained from the equation (8b) to get the time history of central displacement, strain energy and kinetic energy of the shell panel.

4. RESULTS AND DISCUSSIONS

Large amplitude free flexural vibration characteristics of thin isotropic and laminated composite cylindrical panels of constant thickness *h*, radius *R*, axial length *L* and plan-form width *a* (as shown in Fig 1) is considered here. The non-dimensional material properties, unless specified otherwise, used in the present analysis are $E_L/E_T = 40.0$, $G_{LT}/E_T = 0.6$, $G_{TT}/E_T = 0.5$, and $\nu_{LT} = 0.25$, where, *E*, *G*, and ν are Young's modulus, shear modulus and Poisson's ratio respectively. Subscripts *L* and *T* represent the longitudinal and transverse directions with respect to fiber directions. The ply-angles are measured from the circumferential direction to the fiber direction. All the layers are of equal thickness.

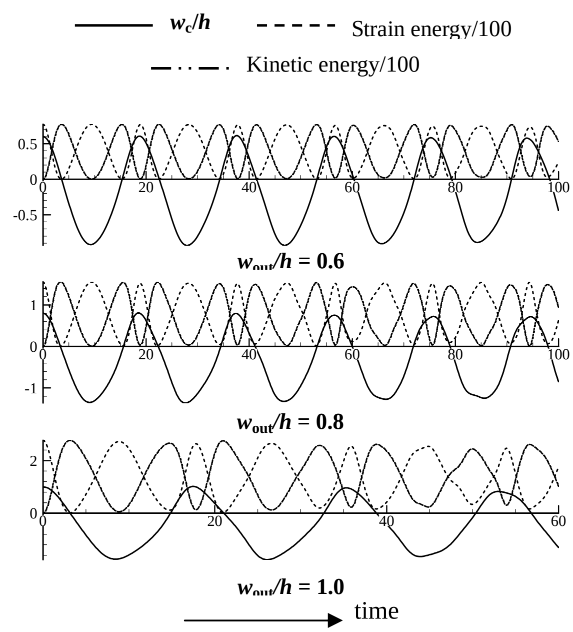


Fig 3. The time history of central displacement (w_c/h), strain energy and kinetic energy of an isotropic simply supported (SS1) cylindrical panel at different amplitudes of vibration. (*R/a* = 10, *a/h* = 100, *L/a* = 1).

The efficacy of the present element for linear free vibration analysis of composite cylindrical panels has been tested earlier [15] and the same is not presented here for the sake of brevity. 16 node MITC shell element has good convergence property for both thin and thick panels and a 4×4 mesh is chosen to model the cylindrical panels. The different boundary conditions considered in the present analysis are:

Simply supported cases:
All edges simply support (SS1): $u_1 = u_2 = u_3 = 0$ along the boundary nodes
Straight edges simply supported, curved edges free (SS2): $u_1 = u_2 = u_3 = 0$ at $x_1 = 0, a$
Clamped case: (CC1)
$u_1 = u_2 = u_3 = \alpha = \beta = 0$ along the boundary nodes

At the beginning, the large amplitude free flexural vibration characteristics of a simply supported (SS1) thin isotropic cylindrical shell panel ($R/a = 10$, $a/h = 100$, $L = a$) are studied from a time history analysis. The nonlinear mode shapes $\{\delta_m, \delta_b\}^T$ are obtained from the matrix-amplitude equation (8b) corresponding to different amplitudes of vibration (w_{out}/h). Thereafter, a dynamic response analysis is performed starting from the initial condition $\delta = \{\delta_m, \delta_b\}^T$ and the variation of non-dimensional central displacement (w_c/h), strain energy and kinetic energy with time is presented in Fig 3. The strain energy and kinetic energy are evaluated using the non-dimensional material properties ($E = 100000.0$ and $\nu = 0.3$) and divided by 100, while plotting with the central displacement (w_c/h) in Fig 3.

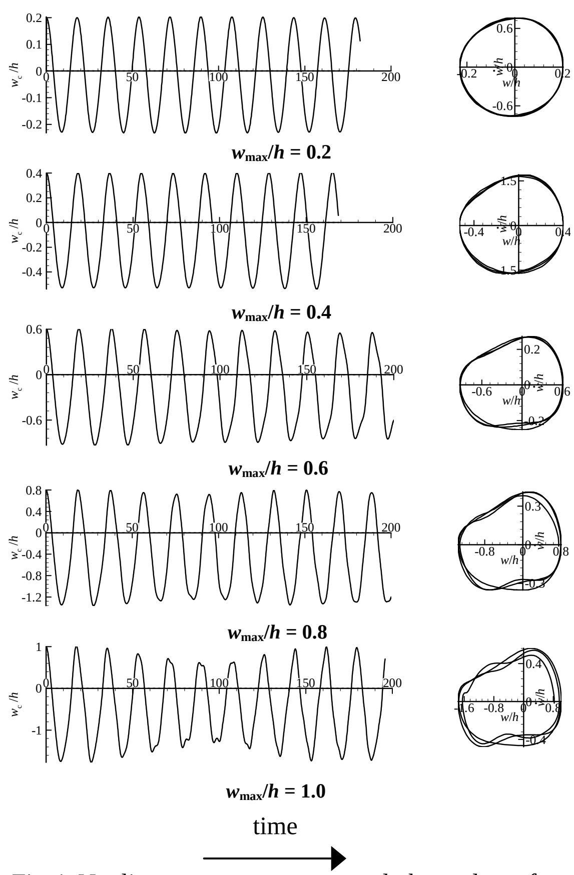


Fig 4. Nonlinear response curves and phase plots of a simply supported (SS1) isotropic cylindrical panel ($R/a = 10$, $a/h = 100$, $L/a = 1$) at different amplitudes of vibration.

From the figure, it is observed that the response is approximately steady-state. In addition, a smooth exchange of the strain energy and kinetic energy (strain energy + kinetic energy is constant) is noticed during the vibration cycle. Hence, it appears that, the vibration mode shapes obtained from equation (8b) is approximately correct, which is further investigated in detail.

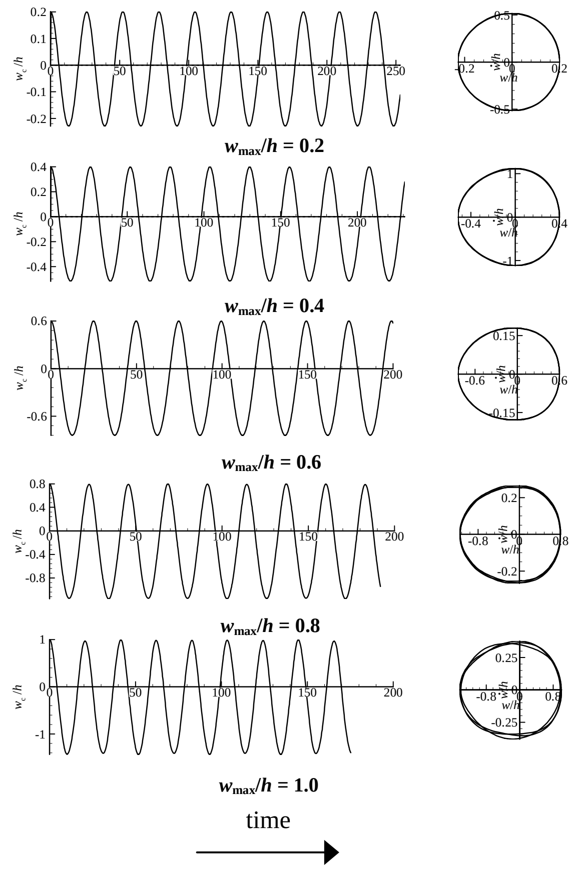


Fig 5. Nonlinear response curves and phase plots of a simply supported (SS1) isotropic cylindrical panel ($R/a = 20$, $a/h = 100$, $L/a = 1$) at different amplitudes of vibration.

The nonlinear dynamic response curves and corresponding phase plots of simply supported (SS1) isotropic cylindrical panels are shown in Fig 4 and Fig 5 for radius-to-thickness ratio (R/a) of 10 and 20 respectively. Time history analysis is also repeated for a cylindrical shell panel ($R/a = 10$) with SS2 boundary condition, (*i.e.*, straight edges simply supported and curved edges are free) and the corresponding dynamic response along with phase plots are reported in Fig 6.

The vibration mode shapes in the outward and corresponding inward directions of an isotropic cylindrical panel ($R/a = 10$, $a/h = 100$) are shown in the Fig 7 and Fig 8 for two different boundary conditions SS1 and SS2 respectively. From the time response curves and the mode shapes, the following major observations are made

- The maximum transverse deflection in the inward direction (towards center of curvature) is more compared to outward displacement (w_{out}/h). The difference between the inward and outward displacements ($w_{in} - w_{out}$) increases with the increase of amplitude of vibration. For example, the inward vibration amplitudes are 0.92*h*, 1.348*h* and 1.747*h* corresponding to the outward displacement of 0.6*h*, 0.8*h* and 1.0*h* respectively for a simply supported (SS1) cylindrical shell panel with $R/a = 10$.

- For low amplitudes of vibration, the mode shape is symmetric and maximum displacement occurs at the center ($w_{max} = w_c$). The vibration response is steady state. Hence, a harmonic time function assumption of the transverse displacement in equation (6) appears to be reasonable.
- With the increase of vibration amplitude, the mode shape becomes un-symmetric with the maximum displacement shifts away from the center ($w_{max} > w_c$). The steady state nature of the vibration gets slowly disturbed. This disturbance is more for the panel with $R = 10a$ compared to the panel with $R = 20a$. Further, the disturbance is more for SS2 boundary condition compared to SS1 boundary condition.

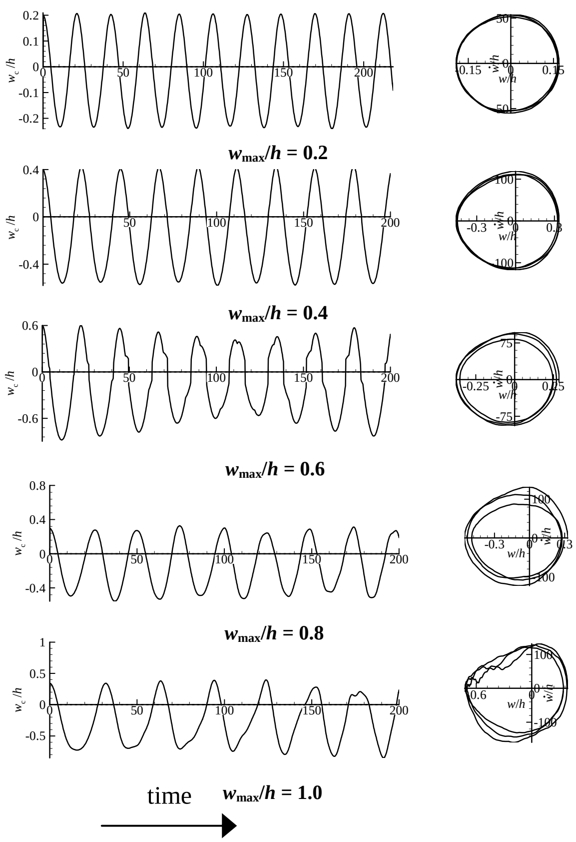


Fig 6. Nonlinear response curves and phase plots of simply supported (SS2) isotropic cylindrical panel ($R/a = 10$, $a/h = 100$, $L/a = 1$) at different amplitudes of vibration.

Next, the variation of maximum transverse displacements in the inward (w_{in}) and outward (w_{out}) directions of thin simply supported (SS1) isotropic cylindrical panels with the total disturbing energy (strain energy + kinetic energy) is studied in Fig 9 for four different values of radius of curvature (R/a = 8, 10, 20 and infinite, *i.e*., plate). It is observed that, the vibration amplitudes (w_{in} and w_{out}) increase with the increase in disturbing energy, followed by a sudden jump in the outward displacements associated with a change in deflection shape

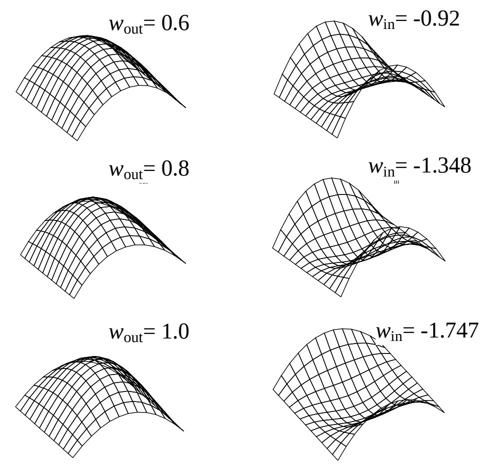


Fig 7. Vibration mode shapes for the outward and the corresponding inward directions of a thin simply supported (SS1) isotropic cylindrical panel ($R/a = 10$, $a/h = 100$, $L/a = 1$).

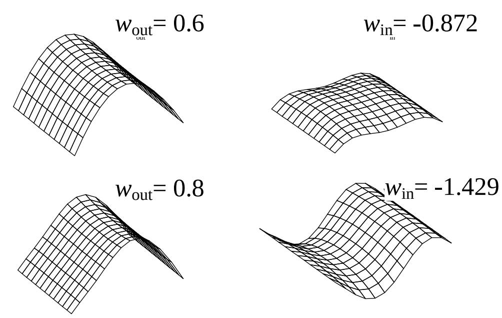


Fig 8. Vibration mode shapes for the outward and the corresponding inward directions of a thin simply supported (SS2) isotropic cylindrical panel ($R/a = 10$, $a/h = 100$, $L/a = 1$).

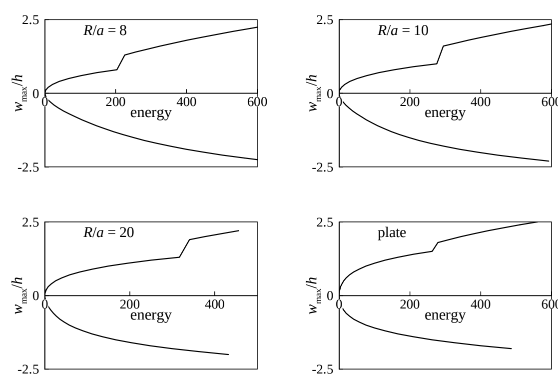
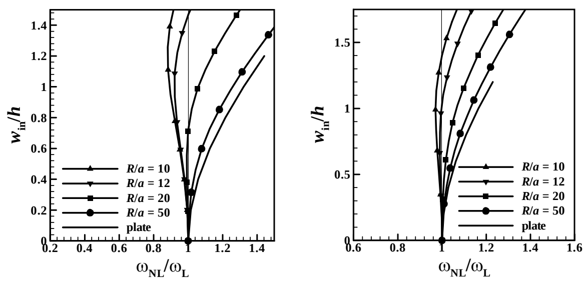


Fig 9. Effect of curvature on the non-linear free vibration characteristics of isotropic simply supported cylindrical panels.

Finally, the nonlinear free vibration behaviors of thin (a/h = 100) angle-ply $[45^0/-45^0/45^0/-45^0/45^0]$ simply supported (SS1) and clamped (CC1) cylindrical shell panels of square plan-form (L/a = 1) are studied in Fig. 10 for different values of radius of curvature (R/a). The nonlinear frequency ratio (ω_{NL}/ω_L) increases with the increase of vibration amplitude (w_{in}/h) for the case of flat plate. This degree of hardening nonlinearity decreases with the increase of curvature (decrease of radius of curvature). For the case of cylindrical panels with *radius-to-span* ratio 10 and 20, the frequency-amplitude relation initially shows a softening behavior. But, at higher amplitudes of vibration, this softening behavior is transformed to hardening behavior. The frequency-amplitude relationship for the clamped cylindrical panel is qualitatively similar to those of simply supported panels.



(a) SS1 (b) CC1

Fig 10. Effect of curvature on the non-linear vibration frequencies of simply supported and clamped (CC1) angle-ply [$45^0/-45^0/45^0/-45^0/45^0$] thin cylindrical panel (a/h = 100, L/a = 1).

5. CONCLUSIONS

Large amplitude flexural vibration characteristics of isotropic and composite cylindrical panels are investigated here using a shear deformable finite element approach. The "*vibration amplitude versus disturbing energy*" and "*vibration amplitude versus nonlinear frequency*" relationships are studied using approximate matrix-amplitude equation obtained from single harmonic approximation to the asymmetric vibration. The present qualitative numerical results are examined thoroughly by a time history analysis. The time history and corresponding phase plots offer a good understanding of the nonlinear vibration characteristics of composite cylindrical panels.

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