Proceedings of the
International Conference on Mechanical Engineering 2009
(ICME2009) 26- 28 December 2009, Dhaka, Bangladesh

ICME09-AM-28

NONLINEAR ANALYSIS OF SHOCK ABSORBERS

M. Ashiqur Rahman, A. Uddin Ahmed, M. Sayem Uddin

Department of Mechanical Engineering,
Bangladesh University of Engineering & Technology, Dhaka

ABSTRACT
Nonlinear dynamics of a two degrees-of-freedom (DOF) shock absorber or, untuned vibration damper system having nonlinear springs and dampers is treated as a boundary value problem. For the different cases, a comparative study is made varying the frequency ratio (r). Another comparison is for response versus time for different spring and damper types at three important frequency ratios: one at $r = 1$, one at $r > 1$ and one at $r < 1$. Response of the system is changed because of the spring and damper nonlinearities; the change is different for different cases. Accordingly, an initially stable absorber may become unstable with time and vice versa. Analysis also shows that higher nonlinearity terms make the system more unstable. Change in response is more evident near the frequency ratio of unity.

Keywords: shock absorber, nonlinear response, initial value problem (IVP), boundary value problem (BVP), multi segment method of integration.

1. INTRODUCTION
Shock absorbers are basically untuned viscous vibration dampers designed to smooth out or damp shock impulse, and dissipate kinetic energy. A tuned vibration absorber is only effective at one frequency, the tuned one, and its usefulness is narrowly limited in the region of the tuned frequency [1]. In contrast, when the forcing frequency varies over a wide range, an untuned viscous damper (also called a shock absorber) becomes very useful [2]. Vibration problems become nonlinear in nature as amplitude of oscillation becomes large in numerous engineering applications [3, 4]. The problem becomes more involved as springs and dampers do not actually behave linearly in vibration problems [4-6]. Zhu et al. [6] extensively studied nonlinear response of two degrees of freedom (2DOF) vibration system with nonlinear damping and nonlinear springs. Recently, Mikhlin and Reshetnikova [5] studied the nonlinear 2DOF system. In papers [7-8], theoretical investigation and some experimental verification on the use of nonlinear localization for reducing the transmitted vibrations in structures subjected to transient base motions have been presented. Nakhaie et al. [9] used the root mean square of absolute acceleration and relative displacement to find the optimal damping ratio and natural frequency of the isolator. Shekhar et al. [10, 11] have considered different alternatives to improve the performance of an isolator having a nonlinear cubic damping over and above the usual viscous damping. Alexander et al. [12] explored the performance of a nonlinear tuned mass damper (NTMD), which is modeled as a TDOFS with a cubic nonlinearity.

Most numerical studies regarding nonlinear vibration of structures, particularly involving multi degrees of freedom system (MDOFS), have been carried out in the form of initial value problems: all the boundary conditions, termed as system's responses (displacement, velocity etc.), were specified at an initial time reference, followed by numerical integration of the governing differential equation. Such type of analysis involves simultaneous solution of a system of nonlinear equations where the number of equations to be solved is determined by order of the governing equations. This type of problem needs to simultaneously solve a large number of nonlinear equations that depends on the number of intermediate grid points in between the two time references. Though, Newton-Raphson method can be used to solve that large number of equations, there are chances of non-convergence of solutions.

Present work aims to solve both boundary and initial value problems for untuned vibration damper system. Generally, stability of the MDOFS system is studied by the method of perturbation. But a simple and direct method, like that of multisegment integration technique [13], that helps to directly visualize the system's response with time, would be very useful, in particular for the present study, when a boundary value problem is dealt with.

Thus results have been obtained for different shock absorbers (Cases 1 – 16 in Table 1) for chosen boundary conditions and different parameters of interest (Tables 2 – 3 and Fig. 1). Ideally, an untuned viscous damper is basically a 2DOFS with a very small mass ratio, having zero damping for the main mass (m_1) and zero spring force for the absorber. Again both the main mass spring and absorber's damper are linear. Such an ideal case has been symbolized as case 1 in this paper. However only to simulate practical situations, a small damping for the main mass (m_1) and a small spring force for the absorber are assumed. It is also assumed that these two nonzero forces can be nonlinear as well (cases 2-16).

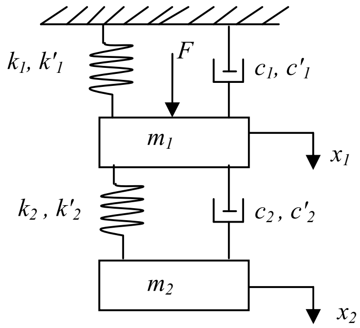

Fig 1. Arrangement of masses, springs and dampers for TDOF vibration system (nonlinear shock absorber).

Table 1: Different cases of shock absorbers

Case	Combinations of Springs	Combinations of Dampers
1	Linear spring with main mass only	Linear damper with second mass only
2	Both springs linear	Linear dampers
3	Both springs hard	
4	1st spring: hard 2nd spring: soft	
5	1st spring: soft 2nd spring: hard	
6	Both springs soft	
7	Both springs linear	Hard dampers
8	Both springs hard	
9	1st spring: hard 2nd spring: soft	
10	1st spring: soft 2nd spring: hard	
11	Both springs soft	
12	Both springs linear	Soft dampers
13	Both springs hard	
14	1st spring: hard 2nd spring: soft	
15	1st spring: soft 2nd spring: hard	
16	Both springs soft	

Table 2: Chosen Parameters of shock absorbers

Parameters	Initial Value Problem	Boundary Value Problem
m_1 (kg)	100.0	100.0
m_2 (kg)	100.0	1.0
k_1 (N/m)	1000.0	1000.0
k'_1(N/m^3)	0.0	±0.5
k_2 (N/m)	0.0	10.0
k'_2 (N/m^3)	0.0	±0.005
c_1 (Ns/m)	0.0	0.03139
c'_1(Ns/m^3)	0.0	±0.003
c_2 (Ns/m)	63.246, 182.174, 316.23, 632.46	3.139
c'_2 (Ns/m^3)	0.0	±0.03
$\sqrt{\frac{k_1}{m_1}}$	3.162	3.162
μ	1.0	0.01
ζ	0.1, ζ_o =0.288, 0.5, 1.0	ζ_o=0.00496
f (N)	20.0	20.0
ω_1 (rad/s)	3.162	3.006
ω_2 (rad/s)	-	3.326

Table 3: Prescribed boundary conditions

Parameters	BVP	IVP
y_1(t=0.0s) (m)	0.05	0
y_2(t=50.0s) (m/s)	0.06	0
y_3(t=0.0s) (m)	-0.07	0
y_4(t=50.0s) (m/s)	-0.06	0

For Table 2, damping constant equals to the optimum damping ratio (ζ_o). These data are so selected to demonstrate the fact that untuned viscous vibration dampers become highly unstable because of increased nonlinearity.

2. MATHEMATICAL MODELS

Fig. 1 shows the proposed model for the TDOFS while Table 1 shows the different cases of shock absorbers. Following Fig. 1, the equations of motion are as follows for the main mass and the damper mass, respectively,

$$m_1\ddot{x}_1 + \left(k_1x_1 + k'_1x_1^3\right) + \left(c_1\dot{x}_1 + c'_1\dot{x}_1x_1^2\right) + \left\{k_2(x_1 - x_2) + k'_2(x_1 - x_2)^3\right\} + \left\{c_2(\dot{x}_1 - \dot{x}_2) + c'_2(\dot{x}_1 - \dot{x}_2)(x_1 - x_2)^2\right\} = F \quad (1)$$

$$m_2\ddot{x}_2 - \left\{k_2(x_1 - x_2) + k'_2(x_1 - x_2)^3\right\} - \left\{c_2(\dot{x}_1 - \dot{x}_2) + c'_2(\dot{x}_1 - \dot{x}_2)(x_1 - x_2)^2\right\} = 0 \quad (2)$$

For transformations, let

$$x_1 = y_1 ,\, x_2 = y_3 \;,\; \frac{dx_1}{dt} = \dot{x}_1 = y_2$$

and $\frac{dx_2}{dt} = \dot{x}_2 = y_4$

With those transformations, Equations 1 and 2 become,

$$m_1\frac{dy_2}{dt} + \left(k_1y_1 + k'_1y_1^3\right) + \left(c_1y_2 + c'_1y_2y_1^2\right) + \left\{k_2(y_1 - y_3) + k'_2(y_1 - y_3)^3\right\} + \left\{c_2(y_2 - y_4) + c'_2(y_2 - y_4)(y_1 - y_3)^2\right\} = F \quad (3)$$

$$m_2 \frac{dy_4}{dt} - \left\{k_2(y_1 - y_3) + k_2'(y_1 - y_3)^3\right\} - \left\{c_2(y_2 - y_4) + c_2'(y_2 - y_4)(y_1 - y_3)^2\right\} = 0 \quad (4)$$

Rearrangement of Equation (3) & (4) gives,

$$\frac{dy_2}{dt} = \frac{1}{m_1}\begin{bmatrix} F - (k_1 y_1 + k_1' y_1^3) - (c_1 y_2 + c_1' y_2 y_1^2) - \left\{k_2(y_1 - y_3) + k_2'(y_1 - y_3)^3\right\} \\ - \left\{c_2(y_2 - y_4) + c_2'(y_2 - y_4)(y_1 - y_3)^2\right\} \end{bmatrix} \quad (5)$$

$$\frac{dy_4}{dt} = \frac{1}{m_2}\left[\left\{k_2(y_1 - y_3) + k_2'(y_1 - y_3)^3\right\} + \left\{c_2(y_2 - y_4) + c_2'(y_2 - y_4)(y_1 - y_3)^2\right\}\right] \quad (6)$$

The governing Equations (5) and (6) can now be rewritten as a set of four nonlinear first order ordinary differential equations (ODE) as follows:

$$\frac{dy_1}{dt} = y_2 \quad (7)$$

$$\frac{dy_2}{dt} = \frac{1}{m_1}\begin{bmatrix} F - (k_1 y_1 + k_1' y_1^3) - (c_1 y_2 + c_1' y_2 y_1^2) - \left\{k_2(y_1 - y_3) + k_2'(y_1 - y_3)^3\right\} \\ - \left\{c_2(y_2 - y_4) + c_2'(y_2 - y_4)(y_1 - y_3)^2\right\} \end{bmatrix} \quad (8)$$

$$\frac{dy_3}{dt} = y_4 \quad (9)$$

$$\frac{dy_4}{dt} = \frac{1}{m_2}\left[\left\{k_2(y_1 - y_3) + k_2'(y_1 - y_3)^3\right\} + \left\{c_2(y_2 - y_4) + c_2'(y_2 - y_4)(y_1 - y_3)^2\right\}\right] \quad (10)$$

The additional field equations, needed for multisegment method of integration are derived now from the above Equations. This is done by differentiating both sides of Governing Equations partially w.r.t. $y(a)$. More details can be found in [1,2].

2.1 Boundary Conditions

For different cases and chosen parameters (Tables 1, 2) prescribed boundary conditions are as given in Table 3. According to the multisegment method of integration, the boundary conditions for any boundary value problem are arranged in the following matrix form,

$$Ay(a) + By(b) = C \quad (11)$$

Solutions of the boundary value problem with any arbitrarily chosen boundary conditions are possible by the present method.

3. RESULTS AND DISCUSSION

Fig. 2 shows the absolute non-dimensional displacement ($|d|$) versus frequency ratio (r) for case 1. This problem is solved as initial value problem. No nonlinearity is taken into consideration. As seen from the figures, peak amplitude is lowest when $\zeta_o = 0.288$ which is known as optimum damping ratio of the system. This analysis was made intentionally to compare the exact results for untuned viscous damper in [4].

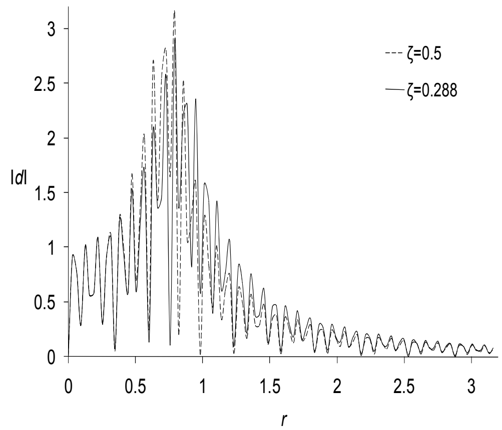

Fig 2. $|d| - r$ for case 1 at t=20s having μ=1.0 and solved as initial value problem ζ=0.288, 0.5.

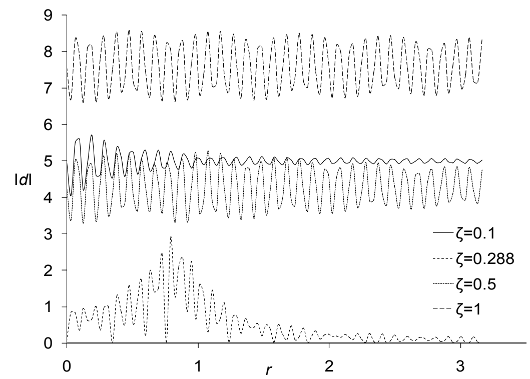

Fig 3. $|d| - r$ for case at t=20s having μ=1.0 and solved as boundary value problem.

These results of Thompson [4] are, of course, for steady state vibrations but present study includes the transient terms. Fig. 2 for the main mass prove the reliability of the code again as these give very similar curves given in Thompson [4].

Next, $d - r$ curves for case 1 having μ=1.0 solved as boundary value problem are shown in Fig. 3. The boundary conditions given in Table 3 were used for this observation. As the boundary conditions were fixing final velocity of the system and damping force is proportional to velocity, eventually the damping force became fixed at that particular boundary (t=b). Damper in this case has little effect on the system's final displacement unless optimum $\zeta = \zeta_o$ is used. As seen from the figure, for ζ=1.0, amplitude (d) range varies from 6.60 – 8.59, for ζ=0.1 range of d varies from 5.0 - 5.7 and for ζ=0.5 d varies from 3.295 to 5.28. But, for optimum damping ratio (ζ=0.288) system shows similar deflection to that of initial value problem. This also indicates that the effect of optimum damping ratio (ζ_o) on the system's response is independent of boundary conditions. This figure also proves the fact that ζ_o gives the minimum d.

$d_{max} - \zeta$ relation for varying mass ratio is shown

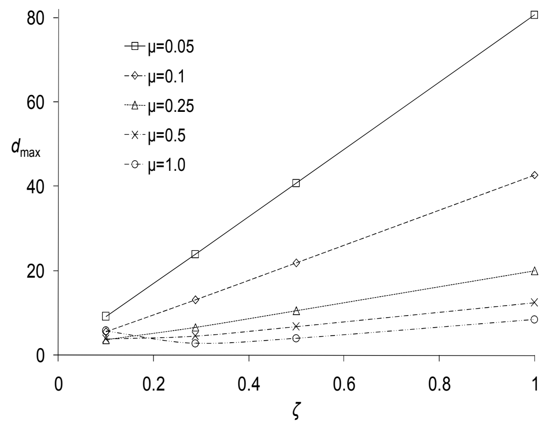

Fig 4. d_{max} - ζ with varying mass ratio (μ) for case 1 and solved as boundary value problem at t=20s.

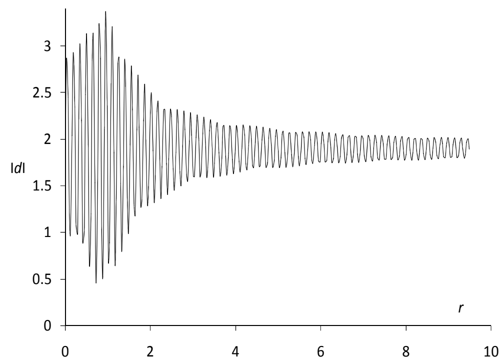

Fig 5. |d| – r for case 2 having μ=0.01and ζ=0.00496 at t=50s and solved as boundary value problem.

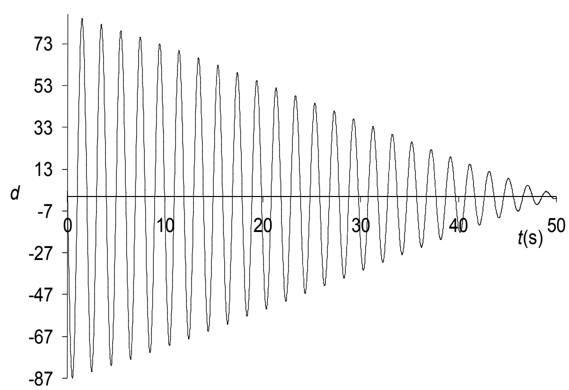

Fig 6. d – t for case 6 with μ=0.01, ζ =0.00496 and r=1.012.

in Fig. 4. Each curve in this figure was drawn by taking the peak amplitudes of a particular mass ratio while varying the damping ratio. From Fig. 4 the line of a particular μ becomes steeper with the decrease of its value. So system with lower mass ratio but with fixed damping ratio gives larger vibration. For cross-check it can be readily seen that for μ=1.0 and ζ=0.288, d_{max} is minimum as discussed earlier.

System response for cases 2 – 16 can be seen in Figs. 5-7.

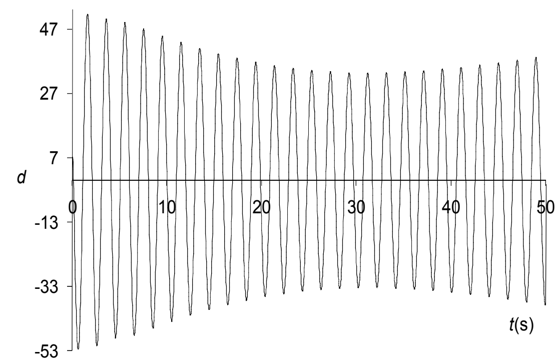

Fig 7. d – t for case 11 with μ=0.01, ζ =0.00496 and r=1.0.

Fig. 5 is for absolute non-dimensional displacement versus frequency ratio for case 2 at t= 50s. At $r \cong 1$ resonance occurs as the system behaves as SDOFS. Due to imposed velocity of the system at final condition the damper force also becomes fixed at the end. As seen from the figure, peak amplitude occurs at frequency ratio near unity and the peak value is 3.37. Cases 3 – 16 show similar type of deflection at t= 50s.

Fig. 6 shows the d – t curves for case 6 at r = 1.012. Case 4 also shows similar type of response at r=1.012. Solution for this type of spring and damper combination, does not converge with time, resulting an unstable system at r=1.0. This happens due to soft springs in absorber side with linear damping (case 4, case 6). Spring force of 2nd mass becomes negative due to negative index and this force cannot be diminished by the linear or soft damper. This problem is eliminated when hard damper is used. Another way to solve this type of problem is to reduce the value of spring index of 2nd mass. As seen from figure, amplitude d ranges from 86.53 – 2.09. Here again system approaches to stability with time.

In Fig. 7, d – t curve for case 11 at r=1.0 is shown. Amplitude ranges from 52.07 – 32.28. Cases 12 – 16 at r=1.0 show similar response. Those cases, not discussed in detail here can be seen in Ahmed [2].

4. CONCLUSIONS

As found in this study case 1 appears to be stable for any value of damping ratio but other cases 2-16 becomes unstable with increase of any nonlinearity index. A few causes of instability can be attributed to jump phenomenon and negative damping for the present study.

Jump phenomenon is for nonlinearity in the spring's response. The amplitude suddenly jumps discontinuously near the resonant frequency making the unstable. The instability region near the resonant frequency depends on amount of damping and rate of change of forcing frequency among others.

Soundness of the code has been demonstrated comparing a few results of present analysis with available exact results. Following conclusions can be drawn from the present study:

- Effect of optimum damping ratio (ζ_o) on the system's response is independent of boundary conditions. Incase of optimum damping, untuned vibration damper vibrates with minimum amplitude. This happens for both initial value and boundary value problems. For example, case 1 with optimum damping ratio (ζ_o=0.288) shows similar deflections for initial value and boundary value problems.
- Mass ratio (μ) plays a significant role on the maximum deflection of untuned vibration damper. For the same damping ratio, system with larger mass ratio shows smaller peak amplitude. But all the systems with a particular mass ratio show minimum peak at optimum damping ratio.
- Increased nonlinearity in spring and damper makes the system more unstable.
- Untuned system with different cases (2 – 16) shows similar responses for both lower and higher forcing frequency ratios. For example, at forcing frequency ratio r =0.3162 and 4.744 all the systems show similar responses.
- From this study, we can conclude that practical springs and dampers should contain smaller nonlinearity indices as systems with lower nonlinearity indexes approached to more stability.
- Practically no spring or damper remains linear with ever-increasing response. Therefore, this study is particularly useful for cars' shock absorber design and application. As stability of a shock absorber can easily change because of damper and spring nonlinearities.

5. REFERENCES

1. Rahman, M. A. and Ahmed, A. U. (2009), "Boundary value problem analysis of a tuned vibration absorber having nonlinear springs", Int. J. Structural Engineering (Inderscience Enterprises Ltd., Switzerland)- In press.
2. Ahmed, A. U., (2009), " Stability analysis of vibration absorbers", M. Sc. Engg. Thesis, Department of Mechanical Engineering, BUET, Bangladesh.
3. Kalnins A. and Dym C. L.,(1976), "Vibration Beams, Plates, and Shells", Dowden, Hutchinson & Ross, Inc.
4. Thomson, W. T., (1981). "Theory of Vibrations with Applications (2nd Edition)", George Allen & Unwin.
5. Mikhlin Y. V., Reshetnikova S. N., (2005). "Dynamical Interaction of an Elastic System and Essentially Nonlinear Absorber", Journal of Sound and Vibration 283 (1-2), pp. 91-120.
6. Zhu S.J., Zheng Y.F., Fu Y.M., (2004). "Analysis of Non-linear Dynamics of a Two Degree of Freedom Vibration System with Non-linear Damping and Nonlinear Spring", Journal of Sound and Vibration 271, pp. 15-24.
7. Manevitch L. I., Gendelman O., Musienko A. I., Vakakis A. F., and Bergman L., (2003), "Dynamic interaction of a semi-infinite linear chain of coupled oscillators with a strongly nonlinear end attachment", Physica D. Nonlinear Phenomena 178(1-2), pp. 1–18.
8. McFarland D. M., Bergman L. A., Vakakis A. F., Manevitch L. I., and Gendelman O., (2002), "Energy pumping into passive nonlinear energy sinks attached to forced linear substructures: analytical and experimental results", 9th Conference on Nonlinear Vibrations, Stability, and Dynamics of Structures, Virginia Polytechnic Institute and State University.
9. Nakhaie G., Narimani A., Golnaraghi M. F., Swanson D.A, (2003), "Practical Frequency and Time Optimal Design of Passive Linear Vibration Isolation Mounts,"
   1. Vehicle System Dynamics 39, pp. 437-466.
10. Shekhar N. C., Hatwal H. , Mallik A. K., (1998), " Response of Non-linear Dissipative Shock Isolators," Journal of Sound and Vibration 214, pp. 589-603.
11. Shekhar N. C., Hatwal H., Mallik A. K., (1999), " Performance of Nonlinear Isolators and Absorbers to Shock Excitation," Journal of Sound and Vibration 227, pp. 293-307.
12. Alexander N. A., Schilder F., (2009), "Exploring the Performance of a Nonlinear Tuned Mass Damper" Journal of Sound and Vibration 319, pp. 445–462.
13. Kalnins, A., and Lestingi, J. E., (1967). "On Nonlinear Analysis of Elastic Shells of Revolution", J. Appl. Mech. 34, pp. 59-64.

6. NOMENCLATURE

a	Initial time reference
b	Final time reference
c	Damping coefficient (N-s/m)
c_1, c_2	Main mass and absorber mass damping coefficients (N-s/m)
c'	Damping nonlinearity index (N-s/m^3)
c'_1, c'_2	Main mass and absorber mass damping nonlinearity indexes (N-s/m^3)
d	k_1*x_1/f : Nondimensional displacement for main mass
e	k_1*x_2/f : Nondimensional displacement for absorber mass
f	Amplitude of the applied force (N)
F	$f\sin(\omega t)$ (N)
k	Spring constant (N/m)
k_1, k_2	Main mass and absorber mass spring constants (N/m)
k'	Spring nonlinearity index (N/m^3)
k'_1, k'_2	Main mass and absorber mass spring nonlinearity indexes (N/m^3)
m_1, m_2	Main mass and absorber mass (kg)
r	Frequency ratio: $\dfrac{\omega}{\sqrt{\dfrac{k_1}{m_1}}}$
r_1	Value of r at first natural frequency of the system
r_2	Value of r at second natural frequency of the system
t	Time (s)
x_1	x
x_1, x_2	Main mass and absorber mass deflections (m)
y_1, y_3	x_1, x_2 (m)
$\dot{x}_1$, $\dot{x}_2$	Main mass and absorber mass velocities (m/s)
y_2, y_4	$\dot{x}_1$, $\dot{x}_2$ (m/s)
ζ	Damping ratio: $\dfrac{c_2}{2\sqrt{m_1k_1}}$
ζ_o	Optimum damping ratio : $\dfrac{\mu}{\sqrt{2(1+\mu)(2+\mu)}}$
μ	Mass ratio: m_2/m_1
ω	Forcing frequency (rad/s)
ω_1,ω_2	Natural frequencies of the main mass and absorber masses
DOF	Degrees of Freedom
DOFS	Degrees of Freedom System
MDOF	Multiple Degrees of Freedom
SDOF	Single Degree of Freedom
Tuned Absorber	2 DOFS without damping
Untuned Absorber	2 DOFS with damping
Spring force	$kx \pm k'x^3$
Damping force	$c\dot{x} \pm c'\dot{x}x^2$
Hard spring	Nonlinearity index (k') is positive
Soft spring	Nonlinearity index (k') is negative
Hard damper	Nonlinearity index (c') is positive
Soft damper	Nonlinearity index (c') is negative
Linear spring	Nonlinearity index k' = 0.0
Linear damper	Nonlinearity index c' = 0.0

7. MAILING ADDRESS

Muhammad Ashiqur Rahman
Professor
Department of Mechanical Engineering,
Bangladesh University of Engineering & Technology (BUET), Dhaka-1000
Telephone & Fax: 88-02-9665636
Email: ashiq@me.buet.ac.bd