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ON THE DISPLACEMENT-POTENTIAL SOLUTION OF PLANE PROBLEMS OF STRUCTURAL MECHANICS WITH MIXED BOUNDARY CONDITIONS

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ABSTRACT

The present paper describes the advancement of displacement-potential approach in relation to solution of plane problems of structural mechanics with mixed mode of boundary conditions. Both the conditions of plane stress and plane strain are considered for analyzing the displacement and stress fields of the structural problem. Using the present approach an elastic plate is solved in which one of its ends is kept fixed and the opposite end is subjected to a combined loading of uniform compression and shear. Finite-difference technique is used to solve the problem numerically. Superiority of the present approach over the usual mathematical models as well as computational approaches is discussed.

Keywords: Elastic analysis, plane problems, displacement potential, finite-difference method.

1. INTRODUCTION

In order to reduce complexities as well as computational effort, a large number of three-dimensional problems are reduced to two-dimensional ones following the standard simplifying assumptions of either plane strain or plane stress condition. If one of the dimensions of a three-dimensional body is larger than other two dimensions, the problem can be considered as plane strain problem. A number of elastic problems have been solved using either plane strain or plane stress conditions based on existing elasticity formulations. Ogden and Isherwood [1] developed new formulations of the governing equations for finite plane-strain deformations of compressible isotropic elastic solids. Shuguang and Lim [2] used variational principles for the generalized plane strain problem of elasticity and they formulated total complementary potential energy functional has been applied to some classic problems in composite materials, viz. the analysis of transversely cracked laminates and the micromechanics of unidirectionally fiber-reinforced composites. Amadei and Goodman [3] developed a more general formulation of plane strain. They applied it to calculate the displacement and stress distributions around a circular hole drilled in a regularly jointed rock described as an equivalent anisotropic continuum. Tewary and Kriz [4] modified elastic plane strain Green's function to account for generalized plane strain and applied to calculating the stress and displacement field in a biomaterial composite containing a free surface normal to the interface and subjected to an out-of-numerically as well as analytically. A generalized plane strain model was employed for the simulations of cold pilgering of fuel cladding for nuclear applications by Harada et al [5]. The difficulties involved in trying to solve the practical stress problems using the existing approaches, for example, the stress function approach or the displacement-functions approach [6], are clearly pointed in Ref. [7]. In the present paper, the displacement potential approach is extended for solving both the categories of plane problems of elasticity. Using this approach, a one-end fixed steel plate subjected to a combined loading of axial compression and shear, at its opposite end, is solved numerically.

2. DISPLACEMENT-POTENTIAL FORMULATION FOR PLANE PROBLEMS OF ELASTICITY

The stress at a point of a two-dimensional elastic body can be represented by three dependent variables, namely, σ_{xx}, σ_{yy} and σ_{xy}. With reference to a rectangular coordinate system and in the absence of body forces, these three variables for the case of isotropic materials are governed by the two equilibrium equations [6].

$$\frac{\partial}{\partial x}\left(\sigma_{xx}\right)+\frac{\partial}{\partial y}\left(\sigma_{xy}\right)=0 \qquad (1)$$

$$\frac{\partial}{\partial y}\left(\sigma_{yy}\right)+\frac{\partial}{\partial x}\left(\sigma_{xy}\right)=0 \qquad (2)$$

To express the equilibrium equations in terms of displacement components, three stress-displacement relations are needed, which are obtained from Hooke's law as [6], are valid for plane stress and plane strain conditions.

$$\sigma_{xx} = \frac{\overline{E}}{1-\overline{\nu}^2}\left[\frac{\partial u_x}{\partial x} + \overline{\nu}\frac{\partial u_y}{\partial y}\right] \qquad (3)$$

$$\sigma_{yy} = \frac{\overline{E}}{1-\overline{\nu}^2}\left[\frac{\partial u_y}{\partial y} + \overline{\nu}\frac{\partial u_x}{\partial x}\right] \qquad (4)$$

$$\sigma_{xy} = \frac{\overline{E}}{2\left(1+\overline{\nu}\right)}\left[\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right] \qquad (5)$$

where $\overline{E} = E$, $\overline{\nu} = \nu$ ……. for plane stress condition

$\overline{E} = \frac{E}{1-\nu^2}$, $\overline{\nu} = \frac{\nu}{1-\nu}$ ….. for plane strain condition

Substituting the above stress-displacement relations into equations (1) and (2), two equilibrium equations for two-dimensional plane problems of isotropic materials in terms of the two displacement components are obtained as [7].

$$\frac{\partial^2 u_x}{\partial x^2} + \frac{1-\overline{\nu}}{2}\frac{\partial^2 u_x}{\partial y^2} + \frac{1+\overline{\nu}}{2}\frac{\partial^2 u_y}{\partial x \partial y} = 0 \qquad (6)$$

$$\frac{\partial^2 u_y}{\partial y^2} + \frac{1-\overline{\nu}}{2}\frac{\partial^2 u_y}{\partial x^2} + \frac{1+\overline{\nu}}{2}\frac{\partial^2 u_x}{\partial x \partial y} = 0 \qquad (7)$$

Although the above two differential equations are theoretically sufficient to solve the mixed-boundary value elastic problems of isotropic materials, in reality it is extremely difficult to solve for two functions simultaneously satisfying the two second-order elliptic partial differential equations. In order to overcome this difficulty, the existence of a new potential function of the space variables is investigated in an attempt to reduce the problem to the determination of a single variable from a single differential equation of equilibrium. In the present formulation, the potential function $\psi\left(x, y\right)$ for plane strain conditions is thus defined in terms of the displacement components as [7]

$$u_x\left(x, y\right) = \frac{\partial^2 \psi}{\partial x \partial y} \qquad (8)$$

$$u_y(x, y) = -\frac{1}{1+\overline{\nu}}\left[2\frac{\partial^2 \psi}{\partial x^2} + \left(1-\overline{\nu}\right)\frac{\partial^2 \psi}{\partial y^2}\right] \qquad (9)$$

Substitution of equations (8) and (9) into equations (6) shows that the equilibrium equation (6) for plane stress and plane strain conditions is automatically satisfied. Therefore, ψ has to satisfy the equilibrium equation (7) only. Expressing equation (7) in terms of the potential function, ψ the condition that ψ has to satisfy for the case of isotropic plane problems becomes [8]

$$\frac{\partial^4 \psi}{\partial x^4} + 2\frac{\partial^4 \psi}{\partial x^2 \partial y^2} + \frac{\partial^4 \psi}{\partial y^4} = 0 \qquad (10)$$

Substituting the expressions as given by equations (8) and (9) into equations (3) to (5), one obtains the explicit expressions of three major stress components in terms of the potential function, ψ , for both the cases of plane stress and plane strain conditions, as follows:

$$\sigma_{xx}\left(x, y\right) = \frac{\overline{E}}{\left(1+\overline{\nu}\right)^2}\left[\frac{\partial^3 \psi}{\partial x^2 \partial y} - \overline{\nu}\frac{\partial^3 \psi}{\partial y^3}\right] \qquad (11)$$

$$\sigma_{yy}\left(x, y\right) = -\frac{\overline{E}}{\left(1+\overline{\nu}\right)^2}\left[\left(2+\overline{\nu}\right)\frac{\partial^3 \psi}{\partial x^2 \partial y} + \frac{\partial^3 \psi}{\partial y^3}\right] \qquad (12)$$

$$\sigma_{xy}\left(x, y\right) = \frac{\overline{E}}{\left(1+\overline{\nu}\right)^2}\left[\overline{\nu}\frac{\partial^3 \psi}{\partial x \partial y^2} - \frac{\partial^3 \psi}{\partial x^3}\right] \qquad (13)$$

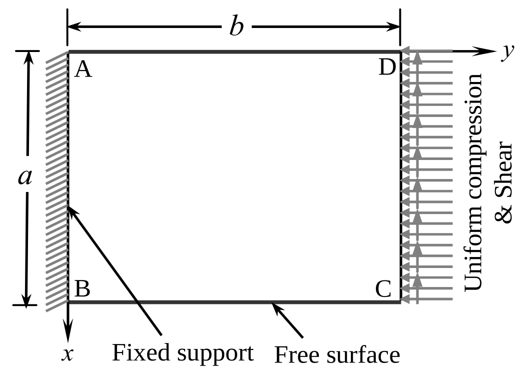

Fig.1 Model of a fixed-ended plate subjected to a combined loading

3. NUMERICAL SOLUTION

Finite-difference technique is used to discretize the governing differential equation (10) and also the differential equations associated with the boundary conditions (Eq. (8), (9), (11) to (13)). The discrete values of the displacement potential function, $\psi(x,y)$, at the mesh points of the domain concerned, is solved from the system of linear algebraic equations resulting from the discretisation of the governing equation of equilibrium and the associated boundary conditions The complete finite-difference expression of the governing equation (10) as well boundary conditions are available in Ref. [8]. Details of the computational scheme as well as management and placement of boundary conditions in terms of finite difference expressions are described in Refs. [7-8].

A square steel plate, ABCD, subjected to a uniform compression and shear loading is shown in Fig. 1. Its supporting edge is AB and a combined loading is applied at its right lateral edge, DC. The intensity of both the components of applied stress is 30 MPa. The Young's' modulus and Poisson's ratio of steel are taken as 200 GPa and 0.3, respectively, to solve the problem.

4. RESULTS AND DISCUSSIONS

In this section, the elastic field of the plate, whose one end is kept fixed and the opposite end is subjected to a combined loading, as shown in Fig. 1, is analyzed with the help of graphical representation. The variation of all displacement and stress components is shown graphically as a function of *x/a* as well as *y*/*b*.

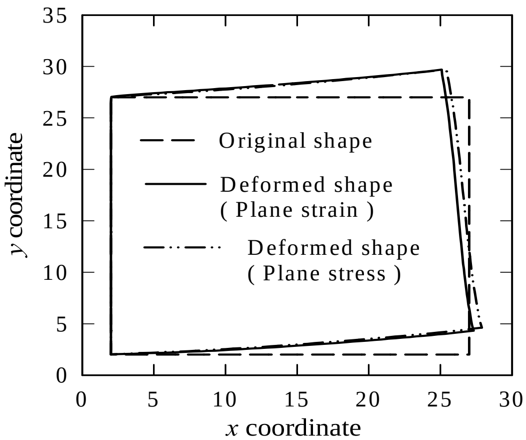

Fig. 2 Deformed shapes of the plate (magnification factor x100)

Figure 2 shows the deformed shapes of the plate for the case of plane stress and plane strain conditions in comparison with the original undeformed configuration. Effect of plane stress as well as plane strain conditions is clearly observed, particularly along the *x*-direction of the plate. At the supporting edge, both of the displacement components are zero, which is in good agreement with the physical characteristics of the problem. The plane strain solution gives a bit lower axial deformation of the plate than that of the plane stress condition. The lateral deformation is also found to be slightly lower for the case of plane strain solution.

The distributions of axial stress component at different sections of the plate are illustrated in a comparative fashion in Fig. 3. It is observed that the magnitude of axial/bending stress is found to be higher for the case of plane stress solution when compared it with the corresponding solutions of plane strain approximation, particularly around the fixed support of

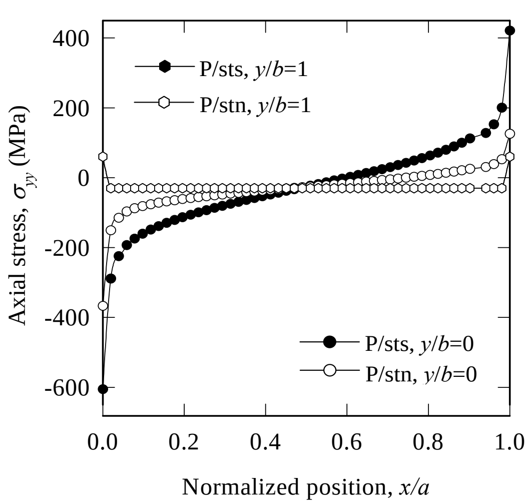

Fig. 3 Distribution of axial stresses at different sections of the plate under different conditions

the plate. The magnitude of the axial stress at the loaded lateral end is found to be identical to that of the applied one, which verifies the accuracy of reproducing the state of stresses at the loaded region. For the remaining longitudinal sections of the plate, the axial stress varies nearly anti-symmetrically with respect to the plate width, for both of the plane stress and plane strain conditions. For almost all sections except the loaded edge of the plate, the overall magnitude of axial stress is observed higher for plane stress condition than that of plane strain condition. Between the two corner regions of the supporting edge, top corner is identified to be more critical than the bottom corner, as far as the axial stress component is concerned.

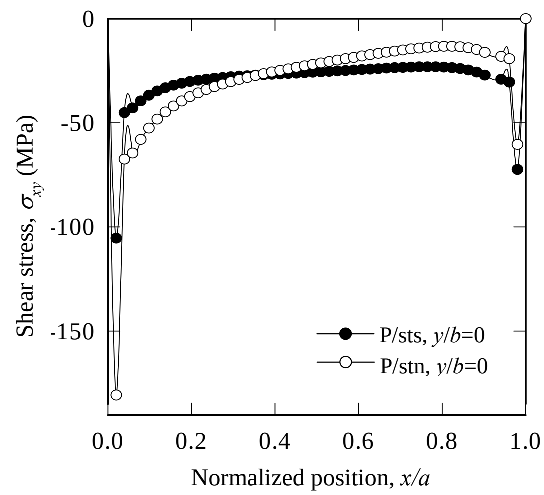

Fig. 4 Distribution of shear stresses at the fixed end of the plate under different conditions

Fig. 4 illustrates the comparison of distributions of shear stress component at different sections of the plate for both the plane stress and plane strain conditions. For both the cases, the shear stress at the right lateral edge of the plate is found identical to that of the applied shear stress, which is again in good agreement to the physical model of the problem. At the supporting end, the effect of the approximation used (*i.e*., plane stress and plane strain conditions) is observed to be quite significant, which is, however, not that significant for the remaining sections of the plate. At the supporting end of the plate, the maximum shear stress under plane strain condition is found to be more than 1.5 times higher than that of the plane stress condition. Likewise the case of axial stress, the upper corner region of the supporting end is identified as the most critical section in terms of shearing stress.

As the lateral displacement in most of the sections of the plate under plane strain condition is lower than that of the plane stress approximation, the body under plane strain is stiffer in lateral direction than that of plane stress condition. If the stiffness characteristic of the plate in different direction is considered, the lateral stress at different sections of the plate under plane strain should be higher than that of plane stress condition, and this phenomenon is reflected in the solution as observed in

Fig. 5. From the comparative analysis of different displacement components at different sections of the plate under the two conditions it is observed that, away from the supporting end, the stiffness variation due to different simplifying approximations becomes more prominent as we move towards the loaded end. 5.

Finally, from the analysis of the normal stress component, σ_{zz}, at different sections of the plate it is observed that, at the loading and supporting ends, the distribution of this stress are found to be highly nonlinear and significant, however, for the remaining sections, the distribution of the normal stress is nearly linear and less significant in terms of magnitude.

5. CONCLUSIONS

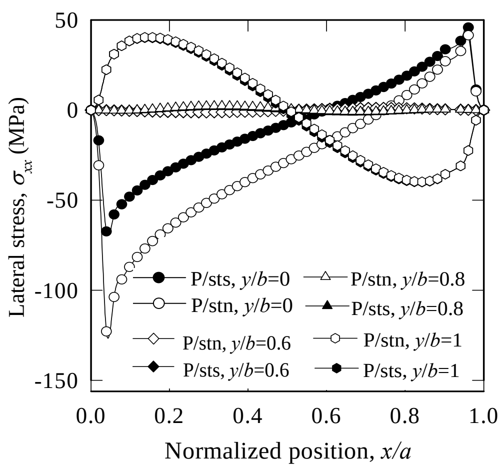

Fig. 5 Distribution of lateral stresses at different sections of the plate under different conditions

In the present article, the displacement-potential approach is extended for the solution of both plane stress and plane strain problems of solid mechanics. A one end fixed plate subjected to a combined loading of uniform compression and shear is solved using finite-difference technique, considering plane stress as well as plane strain conditions. The solutions of the elastic field are presented in a comparative fashion for plane stress and plane strain conditions. The present solution satisfies all the physical characteristics of the problem, which is clearly reflected by the deformed configuration of the plate. Loading and supporting edges of the plate are identified to be highly critical regions for both of the cases, which are realized by the detailed analysis of the results.

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7. NOMENCLATURE

Symbol	Meaning	Unit
u_x	Lateral displacement	(mm)
u_y	Axial displacement	(mm)
σ_{xx}	Lateral stress component	(MPa)
σ_{yy}	Axial stress component	(MPa)
σ_{zz}	Normal stress component	(MPa)
σ_{xy}	Shear stress component	(MPa)
G	Shear modulus of material	(MPa)
E	Elastic modulus of material	(MPa)
ν	Poisson’s ratio	- -
ψ	Displacement potential	
a	Plate width	(mm)
b	Plate length	(mm)

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