Proceedings of the
International Conference on Mechanical Engineering 2009
(ICME2009) 26- 28 December 2009, Dhaka, Bangladesh

ICME09-FM-15

FLOW THROUGH A CURVED RECTANGULAR DUCT

R. Nath Mondal, B. Roy and A. Kumar Datta

Mathematics Discipline; Science, Engineering and Technology School,
Khulna University, Khulna, Bangladesh

ABSTRACT

Flow through a curved rectangular duct of aspect ratios $0.5 \le l \le 1.5$ is investigated numerically by using a spectral and covering a wide range of the Dean number. First, bifurcation structure of the steady solutions is investigated. As a result, a number of steady solution branches with asymmetric two-vortex and symmetric two- and multi-vortex solutions are obtained. The main concern of the present study is to investigate the transitional behavior of the unsteady solutions such as periodic, multi-periodic and chaotic solutions, as the aspect ratio changes. Time evolution calculations as well as their phase spaces show that the steady flow turns into chaotic flow through periodic and multi-periodic flows, if the Dean number is increased no matter what the aspect ratio is. It is found that the flow oscillates periodically or aperiodically between two-, four-, six-, eight- and ten-vortex solutions, as the aspect ratio is increased. It is also found that the axial flow shifted at the outer wall of the duct as the Dean number is increased.

Keywords: Curved Duct, Secondary Flow, Time Evolution, Periodic Solution, Chaos.

1. INTRODUCTION

The study of flows through curved ducts and channels has been and continuous to be an area of paramount interest of many researchers because of the diversity of their practical applications in fluids engineering, such as in fluid transportation, turbo machinery, refrigeration, air conditioning systems, heat exchangers, chemical reactors, ventilators, centrifugal pumps, internal combustion engines and blade- to-blade passage for cooling system in modern gas turbines. Blood flow in the human and other animals also represents an important application of this subject because of the curvatures of many blood vessels, particularly the aorta.

Considering the non-linear nature of the Navier-Stokes equation, the existence of multiple solutions does not come as a surprise. However, an early complete bifurcation study of fully developed flows in a curved duct was conducted by Winters (1987). Yanase *et al.*, (2005) performed numerical investigation of isothermal and non-isothermal flows through a curved duct of rectangular cross-section. Mondal *et al.* (2006) performed numerical prediction of non-isothermal flows through a curved square duct over a wide range of the curvature and the Dean number. Recently, Mondal *et al.* (2007) numerically investigated the bifurcation diagram for two-dimensional steady flow through a curved square duct. Very recently, Mondal *et al.* (2009) performed bifurcation structure of the steady solutions and investigated linear stability of the solutions for the flow through a curved rectangular duct.

Time dependent analysis of fully developed curved duct flows was initiated by Yanase and Nishiyama (1988) for a rectangular cross section. Mondal *et al.* (2007) performed numerical prediction of the solution structure, stability and transitions of isothermal flow through a curved square duct. They showed that there is a close relationship between unsteady solutions and the bifurcation diagram of steady solutions. To the best of the authors' knowledge, however, there has not yet been done any substantial work studying the effects of aspect ratio on unsteady solutions through a curved rectangular duct flows. This paper is, therefore, an attempt, to fill up this gap with a view to study the non-linear nature of the unsteady solutions for various aspect ratios, because this type of flow of often encountered in engineering applications.

In the present study, a numerical result is presented for the fully developed two-dimensional flow of viscous incompressible fluid through a curved rectangular duct. The main objective of the present study is to obtain solution structure of the steady solutions and to discuss the unsteady flow behavior through a curved rectangular channel.

2. MATHEMATICAL FORMULATION

Consider a viscous incompressible fluid streaming through a curved duct with rectangular cross-sections. The coordinate system with relevant notations is shown in Fig. 1. It assumed that the flow is uniform in the z-direction which is driven by a constant pressure gradient G along the centre of the duct. *u*, *v* and *w* are the velocity components in the *x*-, *y*- and *z*-directions, respectively. The variables are non-dimensionalized by using the representative length and the representative velocity.

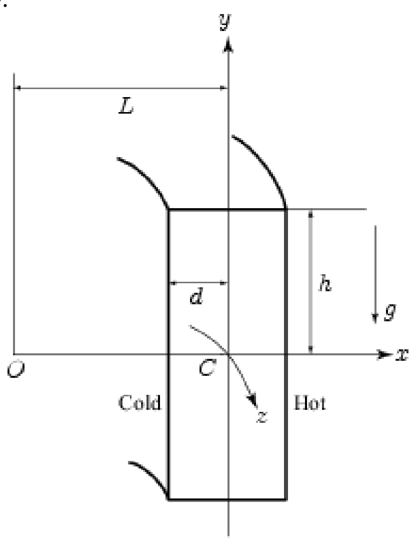


Fig 1. Coordinate system of the curved rectangular duct.

The sectional stream function $\psi(x, y)$ is introduced in the *x*- and *y*- directions as

$$u = \frac{1}{1+\delta x}\frac{\partial \psi}{\partial y}, \qquad v = \frac{1}{1+\delta x}\frac{\partial \psi}{\partial x} \tag{1}$$

Then the basic equations for w *and* ψ are derived from the Navier-Stockes equations as

$$(1+\delta x)\frac{\partial w}{\partial t} + \frac{1}{l}\frac{\partial(w,\psi)}{\partial(x,y)} - Dn + \frac{\delta^2 w}{1+\delta x} = (1+\delta x)\Delta_2 w - \frac{\delta}{l(1+\delta x)}\frac{\partial \psi}{\partial y} w + \delta\frac{\partial w}{\partial x} \tag{2}$$

$$\left(\Delta_2 - \frac{\delta}{1+\delta x}\frac{\partial}{\partial x}\right)\frac{\partial \psi}{\partial t} = -\frac{1}{l(1+\delta x)}\frac{\partial(\Delta_2\psi,\psi)}{\partial(x,y)} + \frac{\delta}{l(1+\delta x)^2}\left[\frac{\partial \psi}{\partial y}\left(2\Delta_2\psi - \frac{3\delta}{1+\delta x}\frac{\partial \psi}{\partial x} + \frac{\partial^2 \psi}{\partial x^2}\right) - \frac{\partial \psi}{\partial x}\frac{\partial^2 \psi}{\partial x \partial y}\right] + \frac{\delta}{(1+\delta x)^2}\left[3\delta\frac{\partial^2 \psi}{\partial x^2} - \frac{3\delta^2}{1+\delta x}\frac{\partial \psi}{\partial x}\right] - \frac{2\delta}{1+\delta x}\frac{\partial}{\partial x}\Delta_2\psi + \frac{1}{l}w\frac{\partial w}{\partial y} + \Delta_2^2\psi \tag{3}$$

where

$$\Delta_2 \equiv \frac{\partial^2}{\partial x^2} + \frac{1}{l^2}\frac{\partial^2}{\partial y^2}, \qquad \frac{\partial(f,g)}{\partial(x,y)} \equiv \frac{\partial f \partial g}{\partial x \partial y} - \frac{\partial f \partial g}{\partial y \partial x}$$

and Dn is Dean number defined as $Dn = \frac{Gd^3}{\mu \nu}\sqrt{\frac{2d}{L}}$, l is the aspect ratio defined as $l = \frac{h}{d}$.

The no-slip boundary conditions for w and ψ are used as:

$$w(\pm 1, y) = w(x, \pm 1) = \psi(\pm 1, y) = \psi(x, \pm 1) = \frac{\partial \psi}{\partial x}(\pm 1, y) = \frac{\partial \psi}{\partial y}(x, \pm 1) = 0 \tag{4}$$

3. NUMERICAL CALCULATION

In order to obtain the numerical solutions, spectral method is used. The main objective of the method is to use the expansion of the polynomial functions that is the variables are expanded in the series of functions consisting of Chebyshev polynomials. The expansion function $\phi_n(x)$ and $\psi_n(x)$ are expressed as

$$\phi_n(x) = (1-x^2)C_n(x), \qquad \psi_n(x) = (1-x^2)^2 C_n(x) \tag{5}$$

where $C_n(x) = \cos(n\cos^{-1}(x))$ is the n^{th} order Chebyshev polynomial. $w(x, y, t)$ and $\psi(x, y, t)$ are expanded in terms of the expansion functions $\phi_n(x)$ and $\psi_n(x)$ as:

$$\left.\begin{aligned} w(x,y,t) &= \sum_{m=0}^{M}\sum_{n=0}^{N} w_{mn}(t)\varphi_m(x)\varphi_n(y) \\ \psi(x,y,t) &= \sum_{m=0}^{M}\sum_{n=0}^{N} \psi_{mn}(t)\psi_m(x)\psi_n(y). \end{aligned}\right\} \tag{6}$$

where M and N are the truncation numbers in the x and y directions respectively. Steady solutions are obtained by the Newton-Raphson iteration method. Finally, for the unsteady solutions, Crank-Nicolson and Adams-Bashforth methods together with the function expansion and collocation methods are applied.

4. RESISTANT CO-EFFICIENT

The resistant coefficient λ is used as the representative quantity of the flow state and is generally used in fluids engineering, defined as

$$\frac{P_1^* - P_{21}^*}{\Delta_{z^*}} = \frac{\lambda}{d_h^*}\frac{1}{2}\rho\langle\omega^*\rangle^2 \tag{7}$$

where quantities with an P_1^* be asterisk denote dimensional ones, $\langle\ \rangle$ stands for the mean over the cross section of the duct and $d_h^* = 4(2d \times 2dl)/(4d + 4dl)$ is the hydraulic diameter. The main axial velocity $\langle\omega^*\rangle$ is calculated by

$$\langle\omega^*\rangle = \frac{\nu}{4\sqrt{2\delta d}}\int_{-1}^{1}dx\int_{-1}^{1}\omega(x,y,t)dy \tag{8}$$

Since $(P_1^* - P_2^*)/\Delta_{z^*} = G,\ \lambda$ is related to the mean non-dimensional axial velocity $\langle \omega \rangle$ as

$$\lambda = \frac{8l\sqrt{2\delta}Dn}{(1+l)\langle\omega\rangle^2} \qquad (9)$$

where $\langle\omega\rangle = \sqrt{2\delta d}\,\langle\omega^*\rangle / \nu$.

5. RESULTS AND DISCUSSION

5.1 Case I: Aspect Ratio L = 0.5

5.1.1 Solution Structure of the Steady Solutions

We obtained a single branch of steady solution for the aspect ratio *l* = 0.5. The solution branch is shown in Fig. 2(a). It is found that the steady solution branch consists of symmetric two- and four-vortex solutions. It is also found that as *Dn* increases, the centrifugal force becomes strong and consequently the axial flow shifted to the outer bend of the wall.

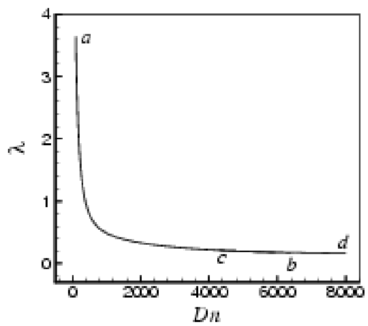
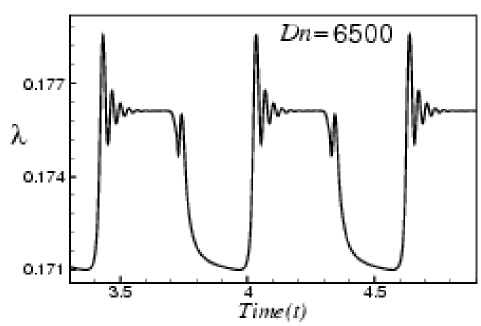


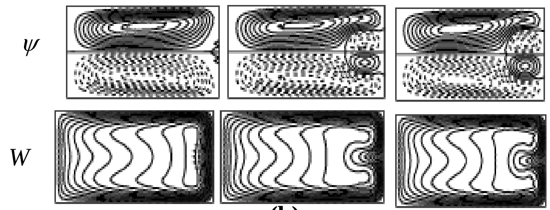
Fig 2. Steady solution branches for the aspect ratio

5.1.2 Unsteady Solutions

To investigate non-linear behavior of the unsteady solutions, time-evolution calculation of the resistance co-efficient λ is performed. Since the steady solution is stable for $Dn \leq 6404$ (Mondal *et al.*, 2009), we performed unsteady solutions for *Dn* > 6404. Time evolution of λ for $Dn = 6500$ is shown in Fig. 3(a), where it is found that the flow is multi-periodic. Then we draw some contours of secondary flow and axial flow distribution in Fig. 3(b), where we observe that the unsteady flow oscillates between symmetric two- and four-vortex solutions.

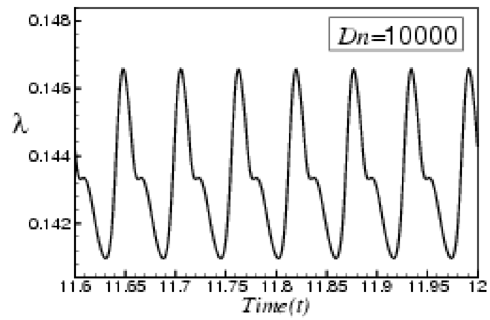


(a)

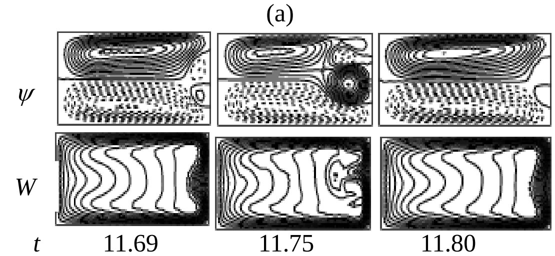


(b)

Fig 3.(a) Time evolution of λ for $Dn = 6500$ and *l* = 0.5. (b) Secondary flow patterns (top) and axial flow distribution (bottom).



(a)



(b)

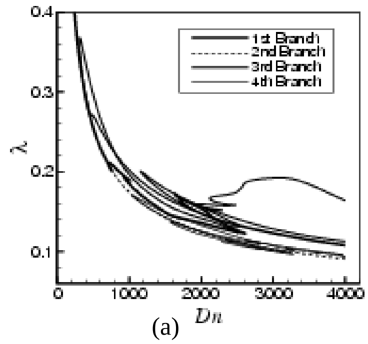
Fig. 4. (a) Time evolution of λ for $Dn = 10000$ and *l* = 0.5. (b) Secondary flow patterns (top) and axial flow distribution (bottom).

Then we performed time evolution of λ for $Dn = 10000$ as shown in Fig 4(a). It is found that the flow is multi-periodic for $Dn = 10000$. Contours of some secondary flow patterns and axial flow distributions are shown in Fig. 4(b), where it is seen that the multi-periodic oscillation at $Dn = 10000$ oscillates between asymmetric two-, three- and four-vortex solutions.

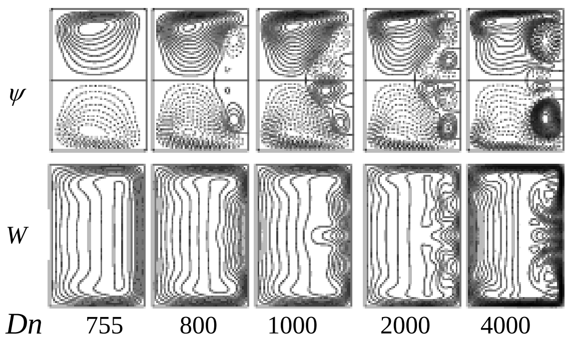
5.2 Case II: Aspect Ratio L = 1.5

5.2.1 Solution Structure of the Steady Solutions

We obtained four branches of steady solutions for the aspect ratio *l* = 1.5. The bifurcation diagram is shown in Fig. 5(a). It is found that there exists a bifurcating relationship between the first and second steady solution branches. The second branch bifurcates from the second branch by a sub-critical pitchfork bifurcation. We obtained two-, four-, six-, eight-, ten- and twelve-vortex solutions on various branches. Figure 5(b) shows contours of some secondary flow patterns and axial flow distribution at various *Dn*.



(a)



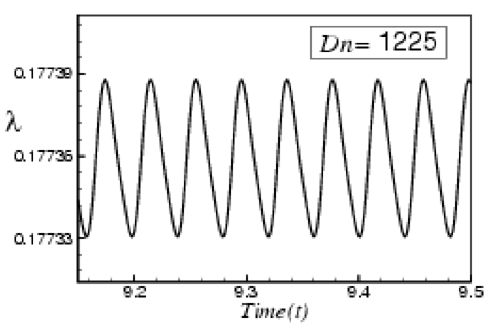
(b)

Fig 5. (a) Bifurcation structure of steady solutions for l = 1.5. (b) Secondary flow patterns (top) and axial flow distribution (bottom) for l = 1.5.

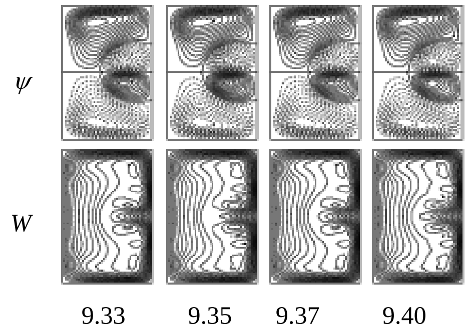
5.2.2 Unsteady Solutions

Unsteady solutions ion is stable for $Dn \leq 625$, we obtained unsteady solutions.

As the steady solut for Dn > 625. We performed time evolution of λ for $Dn = 1225$ as shown in Fig. 6(a). It is found that the flow is time periodic. Contours of secondary flow and axial flow distribution are shown in Fig. 6(b), where it is seen that the periodic solution at $Dn = 1225$ oscillates between symmetric four-vortex solutions.



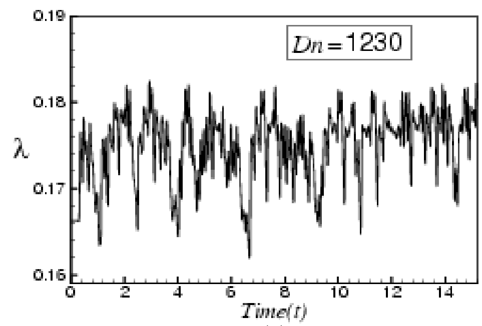
(a)



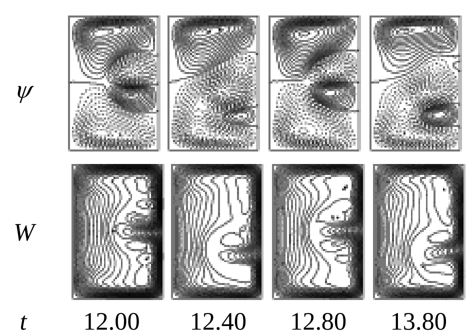
(b)

Fig 6. (a) Time evolution of λ for $Dn = 1225$ and l = 1.5. (b) Secondary flow patterns (top) and axial flow distribution (bottom)

Then we performed unsteady solution for $Dn = 1230$ as shown in Fig. 7(a). It is found that the flow oscillates irregularly, that is the flow is chaotic. Secondary flow patterns and axial flow distribution is shown in Fig. 7(b), where we find that the unsteady flow oscillates between two- and multi-vortex solutions. Since the nature of the flow characteristics changes between Dn = 1225 and Dn = 1230, a transition from periodic to chaotic state occurs between Dn = 1225 and Dn = 1230 for the aspect ratio l = 1.5.



(a)



(b)

Fig 7. (a) Time evolution of λ for $Dn = 1230$ and l = 1.5. (b) Secondary flow patterns (top) and axial flow distribution (bottom).

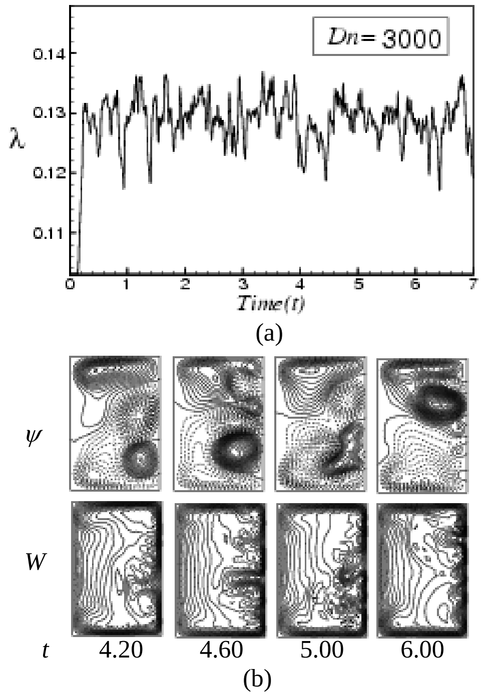


Fig 8. (a) Time evolution of λ for $Dn = 3000$ and l = 1.5. (b) Secondary flow patterns (top) and axial flow distribution (bottom)

Finally, we performed time evolution calculations of λ for $Dn = 3000$ as shown in Fig. 8(a). It is found that the flow is chaotic. Then contours of some secondary flow patterns and axial flow distributions are shown in Fig. 8(b). It is found that the chaotic solution at $Dn = 3000$ oscillates between asymmetric two- and multi-vortex solutions. The chaotic solution at *Dn* = 1230 is called *weak chaos* and that for *Dn* = 3000 *strong chaos* (Mondal *et al*., 2007).

6. CONCLUSIONS

In this study, we obtained solution structure of the steady solutions as well as unsteady solutions for the flow through a curved rectangular duct for the aspect ratios $0.5 \le l \le 1.5$. We obtained a number of steady solution branches with two- and multi-vortex solution on various branches. Time evolution calculation of the unsteady solutions for the aspect ratio 0.5 shows that the flow is periodic or multi-periodic for 6404 <*Dn*< 10000, which oscillates between asymmetric two-, three- and four- vortex solutions. Then we studied time evolution of the unsteady solutions for *Dn* > 625 for the aspect ratio *l* = 1.5, and it is found that the flow is periodic for $Dn = 1225$ but chaotic for $Dn = 1230$. Thus the transition from periodic to chaotic state occurs between *Dn* = 1225 and 1230. We also obtained chaotic solution at large values of the Dean number and it is found that chaotic solution becomes strong at large *Dn*. It is found that as the chaotic solution becomes strong, the number of secondary vortices also increases and the axial velocity shifted at the outer wall of the duct.

7. REFERENCES

1. Mondal, R. N., Kaga, Y., Hyakutake, T. and Yanase, S. (2006). Effects of Curvature and Convective Heat Transfer in Curved Square Duct Flows, *Journal of Fluids Engineering*, Vol. **128**, pp. 1013-1022.
2. Mondal, R. N., Kaga, Y., Hyakutake, T. and Yanase, S. (2007). Bifurcation diagram for two- dimensional steady flow and unsteady solutions in a curved square duct, *Fluid Dynamics Research*, Vol. **39,** pp. 413-446.
3. Mondal, R. N., Uddin, M. S., Ali, M. A. and Datta, A. K. (2009). Laminar flow through a curved duct with rectangular cross section, Bulletin of pure and applied Mathematics, Vol. **3**(1), pp. 55-71
4. Winters, K. H. (1987). A bifurcation study of laminar flow in a curved tube of rectangular cross section, *Journal of Fluid Mechanics*, Vol. **180**, pp. 343-369.
5. Yanase, S. and Nishiyama, K. (1988). On the bifurcation of laminar flows through a curved rectangular tube, *J. Phys. Soc. Japan*, Vol. **57**(11), pp. 3790-3795.
6. Yanase, S. Mondal, R. N., Kaga, Y. and Yamamoto, K. (2005). Transition from Steady to Chaotic States of Isothermal and Non-isothermal Flows through a curved Rectangular Duct, *Journal of the Physical Society of Japan*, Vol. **74(**1), pp. 345-358.

7. MAILING ADDRESS

Rabindra Nath Mondal
Mathematics Discipline;
Science, Engineering and Technology School,
Khulna University, Khulna-9208, Bangladesh
Phone: 0088-01710851580,
Fax: 0088-041-731244
E-mail: rnmondal71@yahoo.com